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Cooperativity and entanglement of atom–field states

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Abstract. The Jaynes–Cummings model of a single two-level atom interacting with a quantized single-mode coherent field generates at the half-revival time a dynamically disentangled atom–field state. At such times, the field is in asymptotically pure Schrödinger cat state, a macroscopic superposition of distinct field eigenmodes. In this paper we address the problem of field purity when a second atom is allowed to interact with the cavity mode and becomes entangled with the first atom via their mutual cavity field with which they interact. We employ the collective Dicke states to describe the cooperative effects on the entanglement and show that the second atom spoils the purity of the field state except for special cases of the atom–field coupling or of initial conditions.

1. Introduction

The interaction of an atom with a quantized electromagnetic field mode leads to an entanglement of these two systems such that the total state vector cannot be written precisely as the product of a time-dependent atomic and field component vector [1]. If the field alone is of interest, the reduced density matrix obtained by tracing over atomic states therefore cannot in general be written as a pure state projector. Yet when the dynamics of the fully quantized Jaynes–Cummings model (JCM) of a two-level atom driven by a single field mode prepared initially in a coherent state [2] is studied in detail, a remarkable *disentanglement* [3] is found at the half-revival time $T_R/2$; the asymptotically-pure field state at this time is approximately a Schrödinger cat superposition [4] of macroscopic states. A number of questions are raised by this result: is this outcome peculiar to a single two-level atom? Would additional dynamically coupled levels, or the presence in the field of more than one atom destroy this disentanglement? Knight and Shore [5] have studied the dynamics of a single multilevel atom interacting with a quantized field mode and show how the disentanglement is highly sensitive to the existence of competing transitions. In this paper we investigate the effect of a field interacting with more than one atom at a time.

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It is a general consequence of entanglement [6] that if a system A with N states becomes entangled with a second system B with M states $(M>N)$, then at most only N states of B (so-called Schmidt states [7] which in general can be orthogonal superpositions over the entire Hilbert space of B, of course) are involved in the dynamics. Thus for a single two-level atom, two such field 'eigenstates' are involved [3]; in the JCM these two manifest themselves in phase space (for example revealed by an appropriate quasiprobability) as a bifurcation into two 'blobs' which separate (in the collapse), recollide (in the revival) in a fashion first revealed by Eiselt and Risken [8]. The essentially pure disentangled state at the half-revival time is a superposition of these two 'blobs' in phase space [3, 4]. A three-level atomic system leads to three field 'eigenstates' and three 'blobs' in phase space [5]; unfortunately as noted by Knight and Shore, they disentangle into a pure state only under very special circumstances when the system can be manipulated into an effective two-level system. In other words, the Schrödinger cat state of the field is essentially poisoned if other levels can participate in the dynamics, and the field reverts to a statistical mixture. One would expect to see similar complications if more than one atom interacts with the field mode, as additional transitions compete for the field excitation.

The traditional JCM describes the interaction of a single two-level atom with a single quantized field mode and leads to a plethora of interesting effects [2] such as the collapse and revival of Rabi oscillations and the vacuum Rabi splitting of spectral lines. The generalization of this model to include several atoms has been studied by a number of authors [9, 10], with particular attention being paid to the collective response of several atoms which see the *same* value of the cavity field. In this case the collective Dicke model of cooperative radiation processes [11] predicts that the Rabi frequency is enhanced by $\sqrt{N}$ where N is the number of atoms in the cavity. Haroche and coworkers [12] and others [13] have exploited this $\sqrt{N}$ dependence to enhance Rabi transients and vacuum Rabi splittings in millimetre-wave Rydberg atom, and high-Q optical cavity experiments. If, however, the spatial variation of the cavity field is important, the collective Dicke model needs to be modified to take into account the fact that each atom in the cavity sees a different value of the radiation field.

In this paper we investigate the evolution of a pair of two-level atoms of variable spatial separation interacting with a single mode quantized field. For small separation compared with the wavelength of the cavity radiation, the evaluation is of course that of the collective *three*-level Dicke model. From the above discussion we would expect three field 'eigenstates' to be involved in the entanglement and three 'blobs' in phase space to emerge, radically different from the two 'blobs' which evolve at $T_R/2$ into a disentangled cat state. However, for modest separations we would expect all *four* Dicke states of the two-atom system, including the antisymmetric state to participate in the dynamics. Under these conditions there are competing pathways [14] involving symmetric and antisymmetric intermediate states to complicate the evolution. The appropriate choice of basis, between Dicke states or simple two-atom products becomes less obvious. Finally, if one of the atoms finds itself situated at a field zero, then of course it cannot participate in the dynamics at all, and the problem reverts to a simple two-level JCM.

One of the surprises that we have found is that for certain initial conditions we find we observe just two 'blobs' in phase space even for an arbitrary distance between the atoms (i.e. even for a four-level system). An analysis of the field density matrix for these cases reveals that under such circumstances we can generate an asymptotically pure state characteristic of a cat once more.

2. Atom–field coupling

Let us consider two two-level atoms interacting with a single mode quantized field. The position of the first atom in the cavity is fixed and the second atom is at some distance R from it. This distance will be a variable parameter of the problem. We have the following Hamiltonian in the dipole and rotating wave approximation (RWA):

$$\hat{H}=\hat{H}_0+\hat{H}_{\text{int}}, \tag{1}$$

where the unperturbed atomic and field ($\hat{H}_0$) and interaction ($\hat{H}_{\text{int}}$) terms are given by

$$\hat{H}_0=\omega_0(\hat{\sigma}_3^{(1)}+\hat{\sigma}_3^{(2)})+\omega\hat{a}^\dagger\hat{a},$$
$$\hat{H}_{\text{int}}=\sum_{i=1,2}\lambda^{(i)}(r)(\hat{\sigma}_+^{(i)}\hat{a}+\hat{a}^\dagger\hat{\sigma}^{(i)}),$$

(we use units such that $\hbar=1$). Here $\hat{\sigma}^{(i)}$ are atomic operators for the ith atom with the usual commutation relations

$$[\hat{\sigma}_+^{(i)},\hat{\sigma}_-^{(i)}]=2\hat{\sigma}_3^{(i)}\delta_{ij},$$

$\hat{a}(\hat{a}^\dagger)$ are field operators corresponding to annihilation (creation) of photons in the cavity mode and obey Bose commutation relations

$$[\hat{a},\hat{a}^\dagger]=1,$$

$\lambda^{(i)}(r)$ is the coupling constant for the ith atom. In the following we will consider the case $\omega_0=\omega$, when atoms and field are exactly resonant. We ignore in this paper the effects of the dipole–dipole interaction on radiative frequency shifts.

The coupling of the first atom is taken to be constant:

$$\lambda^{(1)}(R)\equiv\lambda.$$

Although we take the atom 1 to be situated at a peak of the cavity field mode, we allow the second atom to be at a distance R away and experience a variable atom field coupling

$$\lambda^{(2)}(r)=\lambda\cos kR.$$

The different possible situations are illustrated by figure 1, and are discussed in detail later. Let us note here only that the case $kR=\pi/2$, i.e. $\lambda^{(2)}(r)=0$, corresponds to a conventional two-level JCM, because here the second atom cannot see the field coupling, and the case $kR=0$ corresponds to a three-level two-atom Dicke system. The latter case was treated in [9] with the use of the basis of four states $|g_1,g_2\rangle\otimes|n\rangle$, $|e_1,g_2\rangle\otimes|n-1\rangle$, $|g_1,e_2\otimes|n-1\rangle$, $|e_1,e_2\rangle\otimes|n-2\rangle$, where g_i and e_i correspond to the ground and excited states of the ith atom. But for the case of the Dicke system the triplet and singlet states have independent dynamics. For example, when both atoms are initially excited, only the triplet state evolves. So the essentially three-level Dicke basis is more appropriate for this case and we will

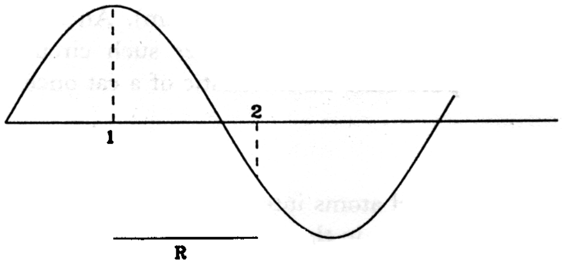


Figure 1. Schematic illustration of the variation of atom–field coupling with position in the cavity mode: atom 1 is fixed at a peak of the cavity field, whereas atom 2 can experience a varying coupling at a distance R away.

use it below. As was stated above, this choice is not so obvious for an arbitrary distance between the atoms when the singlet state is also involved in the dynamics and certainly for the distance such that $kR=\pi/2$, the 'undressed' basis of [9] is more suitable. We employ the following notations for atom–field states:

$$|\varphi_1\rangle=|g_1,g_2\rangle\otimes|n+2\rangle, \quad (3\,a)$$

$$|\varphi_2\rangle=\frac{1}{\sqrt{2}}(|e_1,g_2\rangle+|g_1,e_2\rangle)\otimes|n+1\rangle, \quad (3\,b)$$

$$|\varphi_3\rangle=\frac{1}{\sqrt{2}}(|e_1,g_2\rangle-|g_1,e_2\rangle)\otimes|n+1\rangle, \quad (3\,c)$$

and

$$|\varphi_4\rangle=|e_1,e_2\rangle\otimes|n\rangle. \quad (3\,d)$$

We will work in the Schrödinger picture and write the state of the system in the form

$$|\Psi(t)\rangle=\sum_{i=1}^{4} b_i(t)|\varphi_i\rangle. \quad (4)$$

The Schrödinger equation is written as

$$\mathrm{i}\frac{\mathrm{d}b_k(t)}{\mathrm{d}t}=\omega_0(n+1)+\sum_{j=1}^{4} b_j(t)\langle\varphi_k|\hat{H}_{\mathrm{int}}|\varphi_j\rangle, \quad (5)$$

or, in the matrix form:

$$\frac{\mathrm{d}\mathbf{B}(t)}{\mathrm{d}t}=-\mathrm{i}\mathbf{M}\mathbf{B}(t), \quad (6)$$

where the column vector $\mathbf{B}(t)$ may be written in terms of the probability amplitudes $b_i(t)$

$$\mathbf{B}(t)=\begin{bmatrix} b_1(t)\\ b_2(t)\\ b_3(t)\\ b_4(t)\end{bmatrix}, \quad (7)$$

and the 4×4 matrix **M** contains atom–field energies and spatially-dependent couplings in the form

$$\mathbf{M}=\begin{bmatrix} \omega_0(n+1) & \left(\frac{n+2}{2}\right)^{1/2}\lambda(1+\cos kR) & \left(\frac{n+2}{2}\right)^{1/2}\lambda(1-\cos kR) & 0 \\ \left(\frac{n+2}{2}\right)^{1/2}\lambda(1+\cos kR) & \omega_0(n+1) & 0 & \left(\frac{n+1}{2}\right)^{1/2}\lambda(1+\cos kR) \\ \left(\frac{n+2}{2}\right)^{1/2}\lambda(1-\cos kR) & 0 & \omega_0(n+1) & -\left(\frac{n+1}{2}\right)^{1/2}\lambda(1+\cos kR) \\ 0 & \left(\frac{n+2}{2}\right)^{1/2}\lambda(1+\cos kR) & -\left(\frac{n+1}{2}\right)^{1/2}\lambda(1-\cos kR) & \omega_0(n+1) \end{bmatrix}$$

3. Solutions for the probability amplitudes

The matrix **M** is real and symmetric and has real eigenvalues which we write in the form

$$\nu_i=\omega_0(n+1)+\mu_i\equiv\omega_0(n+1)+\frac{\chi_i}{\sqrt{2}},\qquad i=1,2,3,4, \tag{8}$$

where

$$\chi_{1,2}=[P\pm(P^2-D^2)^{1/2}]^{1/2}, \tag{9 a}$$

and

$$\chi_{3,4}=-[P\pm(P^2-D^2)^{1/2}]^{1/2},$$

with

$$P=(2n+3)(1+\cos^2 kR),\qquad D=2[(n+1)(n+2)]^{1/2}\sin^2 kR. \tag{9 c}$$

Associated with the eigenvalues ν_i we have the following eigenstates (dressed states):

$$\mathbf{B}^{(i)}=\frac{1}{4N_i\chi_i(n+2)^{1/2}\cos kR}\begin{bmatrix} 4\chi_i(n+2)^{1/2}\cos kR \\ -(1+\cos kR)[2(n+2)(1-\cos kR)^2-\chi_i^2] \\ (1-\cos kR)[2(n+2)(1+\cos kR)^2-\chi_i^2] \\ -\frac{\chi_i}{(n+1)^{1/2}}[2(n+2)(1+\cos^2 kR)-\chi_i^2] \end{bmatrix}$$

where N_i are normalization constants ($i=1,2,3,4$).

Let us suppose now that initially both atoms are excited (that is $b_1(0)=b_2(0)=b_3(0)=0$, $b_4(0)=1$) and write down the general solution in the following form:

$$\mathbf{B}(t)=\sum_{j=1}^{4}\alpha^{(j)}\mathbf{B}^{(j)}\exp(-\mathrm{i}\nu_j t). \tag{11}$$

We obtain the following expression for the components of the vector $\mathbf{B}(t)$:

$$b_1(t)=4[(n+1)(n+2)]^{1/2}(\chi_1^2-\chi_2^2)^{-1}(\cos\mu_1 t-\cos\mu_2 t)\exp(-\mathrm{i}at)\cos kR, \tag{12a}$$

$$b_2(t)=\mathrm{i}(n+1)^{1/2}(\chi_1^2-\chi_2^2)^{-1}\left[\frac{2(n+2)(1-\cos kR)^2-\chi_1^2}{\chi_1}\sin\mu_1 t - \frac{2(n+2)(1-\cos kR)^2-\chi_2^2}{\chi_2}\sin\mu_2 t\right]\exp(-\mathrm{i}at)(1+\cos kR), \tag{12b}$$

$$b_3(t)=-\mathrm{i}(n+1)^{1/2}(\chi_1^2-\chi_2^2)^{-1}\left[\frac{2(n+2)(1+\cos kR)^2-\chi_1^2}{\chi_1}\sin\mu_1 t - \frac{2(n+2)(1+\cos kR)^2-\chi_2^2}{\chi_2}\sin\mu_2 t\right]\exp(-\mathrm{i}at)(1-\cos kR), \tag{12c}$$

$$b_4(t)=-(\chi_1^2-\chi_2^2)^{-1}\{[2(n+2)(1+\cos kR)^2-\chi_1^2]\cos\mu_1 t - [2(n+2)(1+\cos kR)^2-\chi_2^2]\cos\mu_2 t\}\exp(-\mathrm{i}at), \tag{12d}$$

where

$$a\equiv\omega_0(n+1). \tag{13}$$

The functions $|b_i(t)|^2$ give us the probabilities of occupation of the ith Dicke state as a function of time, photon occupation number and interatomic distance. It can be seen clearly from equation (12) that depending on R we have some particular cases which we now consider in detail.

3.1. (*A*) $kR=0$

Both atoms are at the same point, so we have a three-level two-atom Dicke system. In this case both atoms see identical fields, hence the dipole transition couplings for both atoms are equal, and the antisymmetric state decouples from the dynamics and this is manifested by $b_3(t)=0$. The state $|\varphi_3\rangle$ is not occupied at any moment of time and we have throughout the evolution of a three-level system. The situation is illustrated in figure 2 (*a*). In this case equation (12) reduces to

$$b_1(t)=[(n+1)(n+2)]^{1/2}(2n+3)^{-1}\{\cos\lambda t[2(2n+3)]^{1/2}-1\}\exp(-\mathrm{i}at), \tag{14a}$$

$$b_2(t)=-\mathrm{i}\left(\frac{n+1}{2n+3}\right)^{1/2}\sin\lambda t[2(2n+3)]^{1/2}\exp(-\mathrm{i}at), \tag{14b}$$

$$b_3(t)\equiv 0, \tag{14c}$$

and

$$b_4(t)=(2n+3)^{-1}\{(n+2)+(n+1)\cos(\lambda t[2(2n+3)]^{1/2})\}\exp(-\mathrm{i}at). \tag{14d}$$

Our result for the probability amplitude for the state $|\varphi_4\rangle$ agrees with that found in [9], where only the $R\to 0$ Dicke limit was studied.

3.2. (*B*) $kR=\pi$

In this case both atoms couple to the field but with opposite signs (see figure 1). The antisymmetric state decouples and we have exactly the same equations as in the previous case with the major difference being that $b_2(t)$ and $b_3(t)$ are interchanged. The scheme of transitions is shown in figure 2(*b*).

3.3. (*C*) $kR=\pi/2$

With this choice of interatomic distance we find that the amplitude $b_1(t)$ is equal to zero at all times, which means that the state $|\varphi_1\rangle$, where both atoms are in their ground states, cannot be populated during the evolution. We can see the reason for this in figure 1. The second atom is located at the point with a zero field amplitude, so it does not see the field at all and does not participate in the evolution. The system is effectively described in terms just of two levels. For this case the basis (2) is not the most suitable and it is better to work with that of individual atoms used in [9] as we can see from figure 2 (*c*). In the product basis of two-level atoms the dynamics is of course that of a two-level atom; in the Dicke basis we need three levels and destructive interference reconstructs the usual two-level JCM evolution.

3.4. (*D*) $kR=\pi/4$

This case is chosen as an intermediate one. Now all the four states are involved in the evolution (figure 2 (*d*)). The new feature of this level scheme is the existence of two pathways by which the fully excited two-atom system can evolve to the state in which both atoms are unexcited. As we shall see, these pathways interfere and this interference modifies the evolution.

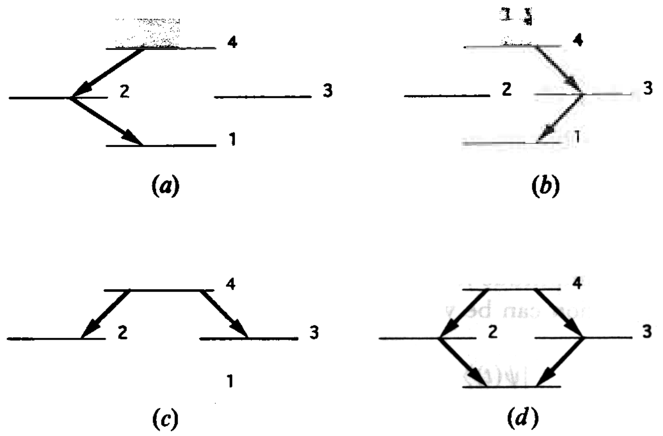


Figure 2. Level scheme for two atoms for different values of kR. (*a*) $kR=0$, we have four Dicke states 1–4, the antisymmetric state 3 decouples and only 4,2 and 1 participate in the dynamics; (*b*) $kR=\pi$; it is the symmetric Dicke state which is now decoupled from the dynamics; (*c*) $kR=\pi/2$; atom 2 now sits at the zero of the cavity field and cannot participate in the collective dynamics, leading to a lambda configuration of Dicke states; (*d*) level configuration appropriate to an arbitrary interatomic distance; all four Dicke states participate in the dynamics, leading to an interesting competition between the 4-2-1 and 4-3-1 transition pathways.

4. Evolution of atomic and field observables

Let us now investigate some atomic and field properties using equations (12). In order to do so we write down the expression for the atom–field system density matrix with the field initially prepared in the number state $|n\rangle$. We have

$$\rho(t)=|\psi(t)\rangle\langle\psi(t)|=\sum_{i=1}^{4}\sum_{j=1}^{4} b_i(t)b_j^*(t)|\varphi_i\rangle\langle\varphi_j|. \tag{15}$$

To obtain the reduced atomic density matrix we have to trace out the field variables

$$\rho_A(t)=\mathrm{Tr}_F\,\rho(t).$$

The density matrices for each atom can be written as

$$\begin{aligned}\rho_1(t)&=\mathrm{Tr}_2\,\rho_A(t)\\ &=\left[|b_1(t)|^2+\frac{1}{2}|b_2(t)-b_3(t)|^2\right]|g_1\rangle\langle g_1|\\ &\quad+\left[\frac{1}{2}|b_2(t)+b_3(t)|^2+|b_4(t)|^2\right]|e_1\rangle\langle e_1|,\end{aligned} \tag{16a}$$

$$\begin{aligned}\rho_2(t)&=\mathrm{Tr}_1\,\rho_A(t)\\ &=\left[|b_1(t)|^2+\frac{1}{2}|b_2(t)+b_3(t)|^2\right]|g_2\rangle\langle g_2|\\ &\quad+\left[\frac{1}{2}|b_2(t)-b_3(t)|^2+|b_4(t)|^2\right]|e_2\rangle\langle e_2|\end{aligned} \tag{16b}$$

We obtain the following reduced field density matrix for this initial Fock field state:

$$\rho_F(t)=|b_1(t)|^2|n+2\rangle\langle n+2|+[|b_2(t)|^2+|b_3(t)|^2]|n+1\rangle\langle n+1|+|b_4(t)|^2|n\rangle\langle n| \tag{17}$$

If we start with an arbitrary initial state of the field $|\kappa\rangle$ (providing it is a pure state)

$$|\kappa\rangle=\sum_{n=0}^{\infty} C_n(\kappa)|n\rangle, \tag{18}$$

where $C_n(\kappa)$ is the amplitude for the state $|\kappa\rangle$ to be occupied initially by n photons, then the wavefunction can be written as

$$|\psi(t)\rangle=\sum_{i=1}^{4}\sum_{n=0}^{\infty} C_n(\kappa)b_{i,n}(t)|\varphi_i(n)\rangle. \tag{19}$$

Here we have added the index n to the notations for $b_1(t)$ and $|\varphi\rangle$, as a reminder that these amplitudes describe atom and field states with a particular cavity-field excitation number.

The dependence of the density matrix on the field photon number distribution is given by

$$\rho(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{4}\sum_{j=1}^{4} C_n(\kappa)C_m^*(\kappa)b_{i,n}(t)b_{j,m}^*(t)|\varphi_i(n)\rangle\langle\varphi_j(m)|. \tag{20}$$

We consider the case when the field is initially in the coherent state $|\alpha\rangle$, i.e. where the occupation amplitude is given by the Poisson amplitude

$$C_n(\kappa)\equiv C_n(\alpha)=\frac{\alpha^n}{\sqrt{n!}}\exp(-|\alpha|^2/2), \tag{21}$$

and $|\alpha|^2=\bar{n}$ is the mean number of photons initially in the cavity field.

We will plot the population inversions for each atom

$$W_i(t)=\frac{1}{2}\mathrm{Tr}\{[|e_i\rangle\langle e_i|-|g_i\rangle\langle g_i|]\rho_i(t), \qquad i=1,2,$$

$$W_1(t)=\frac{1}{2}\sum_{n=0}^{\infty}[|b_{4,n}(t)|^2-|b_{1,n}(t)|^2+b_{2,n}(t)b^*_{3,n}(t)+b^*_{2,n}(t)b_{3,n}(t)]|C_n(\alpha)|^2,$$

and

$$W_2(t)=\frac{1}{2}\sum_{n=0}^{\infty}[|b_{4,n}(t)|^2-|b_{1,n}(t)|^2-b_{2,n}(t)b^*_{3,n}(t)-b^*_{2,n}(t)b_{3,n}(t)]|C_n(\alpha)|^2,$$

and also the total population inversion $W(t)=\frac{1}{2}[W_1(t)+W_2(t)]$:

$$W(t)=\frac{1}{2}\sum_{n=0}^{\infty}[|b_{4,n}(t)|^2-|b_{1,n}(t)|^2]|C_n(\alpha)|^2.$$

The population inversions clearly exhibit the characteristic JCM collapses and revivals and provide us with information about the discrete nature of the quantized atom-cavity field eigenvalues. To obtain a clearer view of the Schmidt eigenmodes of the field we also compute the relevant field quasiprobabilities in phase space. These clearly display, in a series of bifurcations and collisions, the role of such eigenmodes. In particular, we will investigate the Q function, defined by $Q(\beta)=(1/\pi)\langle\beta|\rho_F(t)|\beta\rangle$ ($|\beta\rangle$ is a coherent state) computed for the field reduced density matrix

$$\begin{aligned}\rho_F(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|n\rangle\langle m|\\ \times\{b_{1,n}(t)b^*_{1,m}(t)C_n(\alpha)C^*_m(\alpha)+b_{4,n+2}(t)b^*_{4,n+2}(t)C_{n+2}(\alpha)C^*_{m+2}(\alpha)\\ +[b_{2,n+1}(t)b^*_{2,m+1}(t)+b_{3,n+1}(t)b^*_{3,m+1}(t)]C_{n+1}(\alpha)C^*_{m+1}(\alpha)\},\end{aligned} \tag{25}$$

and obtain the quasiprobability in terms of probability amplitudes and photon occupation amplitudes in the form

$$\begin{aligned}Q(\beta)=\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}C^*_n(\beta)C_m(\beta)\{b_{1,n}(t)b^*_{1,m}(t)C_n(\alpha)C^*_m(\alpha)\\ +[b_{2,n-1}(t)b^*_{2,m+1}(t)+b_{3,n+1}(t)b^*_{3,m+1}(t)]C_{n+1}(\alpha)C^*_{m+1}(\alpha)\\ +b_{4,n+2}(t)b^*_{4,n+2}(t)C_{n+2}(\alpha)C^*_{m+2}(\alpha)\}.\end{aligned} \tag{26}$$

The Q function tells us the number and dynamics of the field eigenmodes. It does not tell us whether these eigenmodes govern the evolution in a pure state or mixed state at particular times. To do this, we employ (following Gea-Banacloche [3]) the purity factor, defined as $\xi(t)=1-\mathrm{Tr}\,[\rho_F^2(t)]$. This parameter characterizes the purity of the field state. For the pure states it must be equal to zero, and deviations from zero indicate the extent to which the field is in a statistical mixture.

We will start the discussion of the results in graphical form with the case $kR=\pi/2$, corresponding to the conventional two-level Jaynes-Cummings model [2]. In figure 3 we observe a sequence of collapses and revivals for the inversions and in figure 4 for the level occupations. The Q function for this case, see figure 5 (*a–c*), bifurcates into two blobs rotating in the complex β plane in the clockwise and counterclockwise directions with the same speed. The collision of the blobs at the point $(-\alpha, 0)$ corresponds to the first revival, the next collision at the point $(\alpha, 0)$, to the second revival [3, 4]. At the moment corresponding to one half of the revival time the field can be described as a superposition of two macroscopically

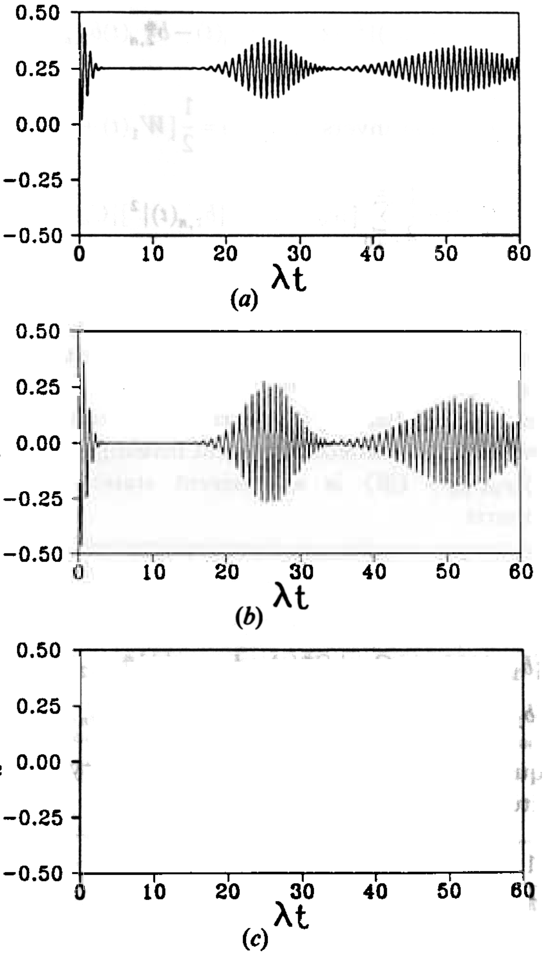


Figure 3. Time evolution of the total atomic inversion defined by equation (24) and inversions of atoms 1 and 2 defined by (22) and (23) for interatomic distance $kR=\pi/2$ starting with both atoms excited and the field in a coherent state with a mean photon number $\bar{n}=16$.

distinguishable pure states, the so-called cat state [4]. This approach to a cat state is manifested by the value of the purity factor which is close to zero (figure 6) at the half-revival time.

Let us discuss now the three-level Dicke case of two identical atoms seeing precisely the same field ($kR=0$). We observe the series of collapses and revivals again (as seen for example by Deng and by Iqbal *et al.* [9]), but there are notable differences from the case $kR=\pi/2$. These can be clearly seen if we compare the graphs for inversions (figure 7), level occupations (figure 8) and Q function (figure 9). The Q function allows us to make a clear physical interpretation of the collapse–revival sequence. We now observe a stationary blob (seen by Knight and Shore [5] for a closely related three-level problem) and two moving blobs. The

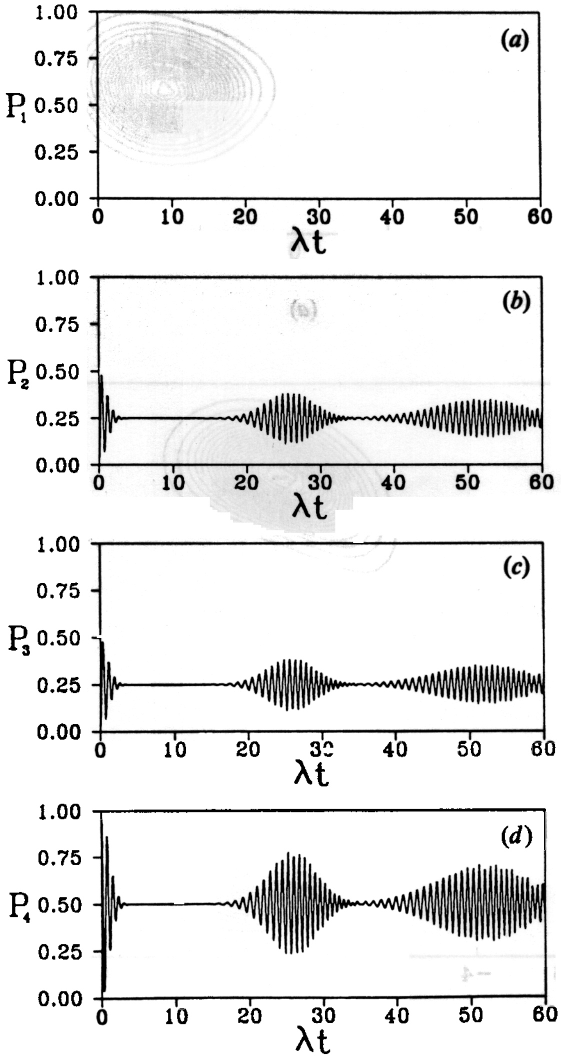
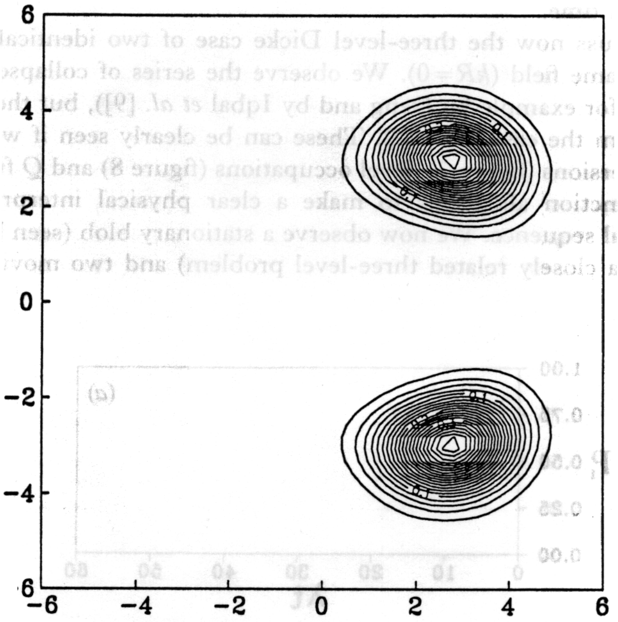
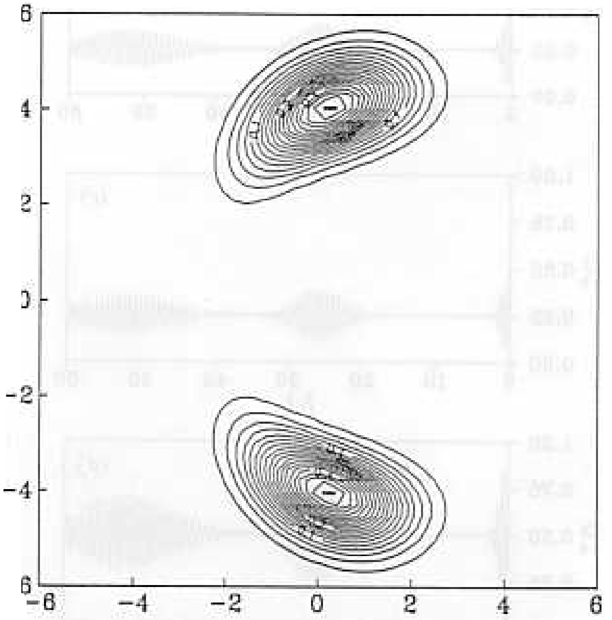


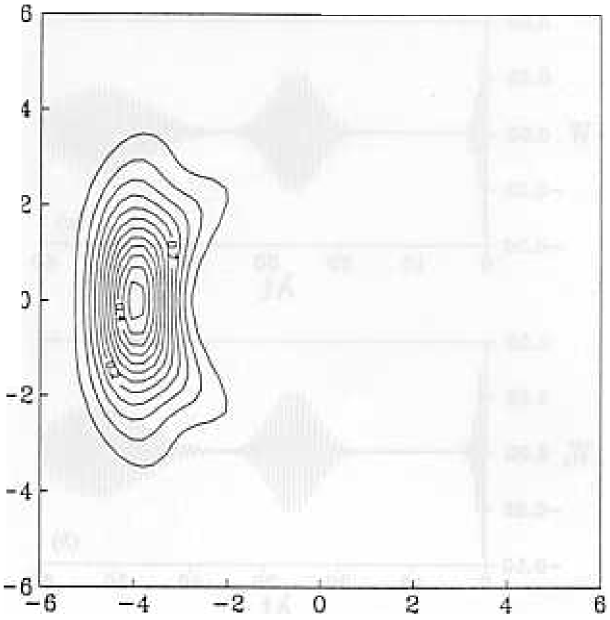
Figure 4. Time evolution of level occupations $P_i(t)$ for $kR=\pi/2$, both atoms are initially excited.



(*a*)



(*b*)



(*c*)

Figure 5. Contour plot of the field reduced density matrix Q function quasiprobability for an interatomic distance $kR=\pi/2$ with $\bar{n}=16$ for different moments of time: (*a*) $\lambda t=6\cdot25$; (*b*) $\lambda t=12\cdot5$; (*c*) $\lambda t=25$. Both atoms are initially excited.

first revival for P_i corresponds to the collision of the moving blobs in the left half-plane and the first one for W_i to the collision of the moving blobs with the stationary one in the right half-plane. So we have different sequences of collapses and revivals for level occupations and inversions. Moreover, we have practically the same pictures for inversions as in the previous case $kR=\pi/2$. This means that the inversions W_1 are identical for the two atom and one atom cases, so in our case

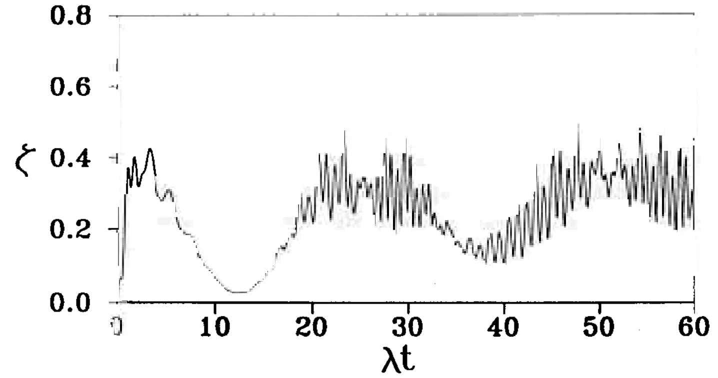


Figure 6. Field purity ξ as a function of λt; $kR=\pi/2$, $\bar{n}=16$. Both atoms are initially excited.

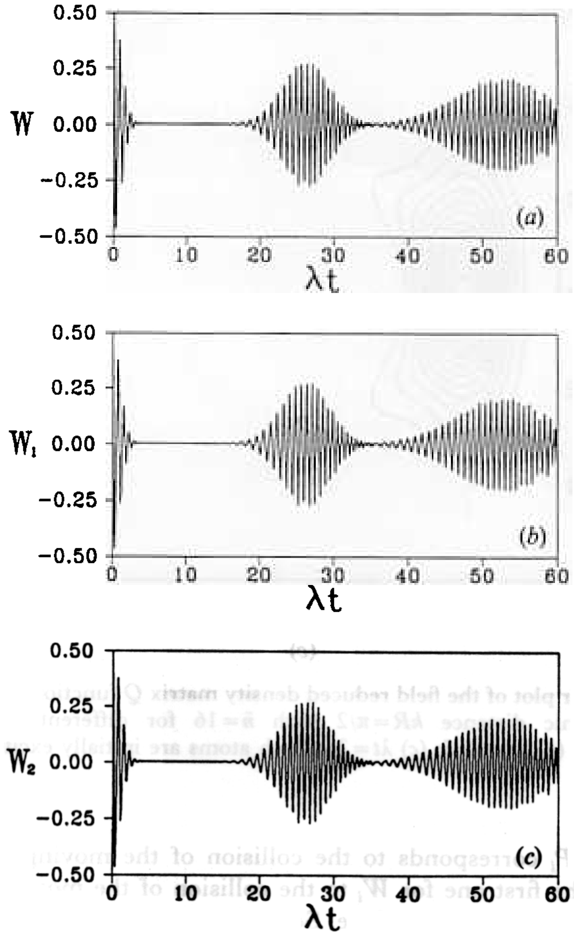


Figure 7. As figure 3, but for $kR=0$ showing a three-level Dicke evolution.

of indirect coupling between the atoms it *seems* that one atom does not feel the presence of the other. We note that the graphs are evaluated using a mean photon number $\bar{n}=16$, i.e. to a rather strong field by JCM standards. In that case the semiclassical approximation is valid, so the above result is in agreement with [15], where independence of the inversion on the number of atoms in semiclassical approximation was predicted. The purity factor (figure 10) for the half revival time is not so close to zero, so the almost pure state of the previous case is spoilt, and at such distances we do not expect the dynamical creation of a Schrödinger cat state, essentially due to the existence of the stationary eigenmode.

For the case $kR=\pi$ we obtain exactly the same pictures as figure 7–10 providing we exchange $b_2(t)$ for $b_3(t)$.

If we consider the four-level system ($kR=\pi/4$) the pattern of collapses and revivals is much more complicated (and again relate to results obtained by Knight and Shore [5] for a related four-level model). Now we have four blobs for the Q function (figure 11): two fast and two slow, moving in pairs in the clockwise and counterclockwise directions on a circle with the radius $(\bar{n})^{1/2}$. Collisions between the blobs of the same type and of different types are possible. We can see from the pictures for level occupations (figure 12) and inversions (figure 13) that any collision of the blobs leads to revival of P_1 and P_4. Collision of the fast blobs corresponds to a revival of P_2, whereas collision of the slow blobs, to the revival of P_3. If fast blobs and slow blobs collide moving in the opposite directions, there is a revival of W_1. If they collide moving in the same direction, there is a revival of W_2. The time dependence of the purity factor (figure 14) is now very complicated, and the field is far from a pure state throughout the evolution.

Let us remark here that a change of initial conditions leads to changes in the pictures, e.g. the structure of the Q function can be different. If we take the state $|\varphi_2\rangle$ to be initially occupied, we obtain the following system of equations instead of equation (4)

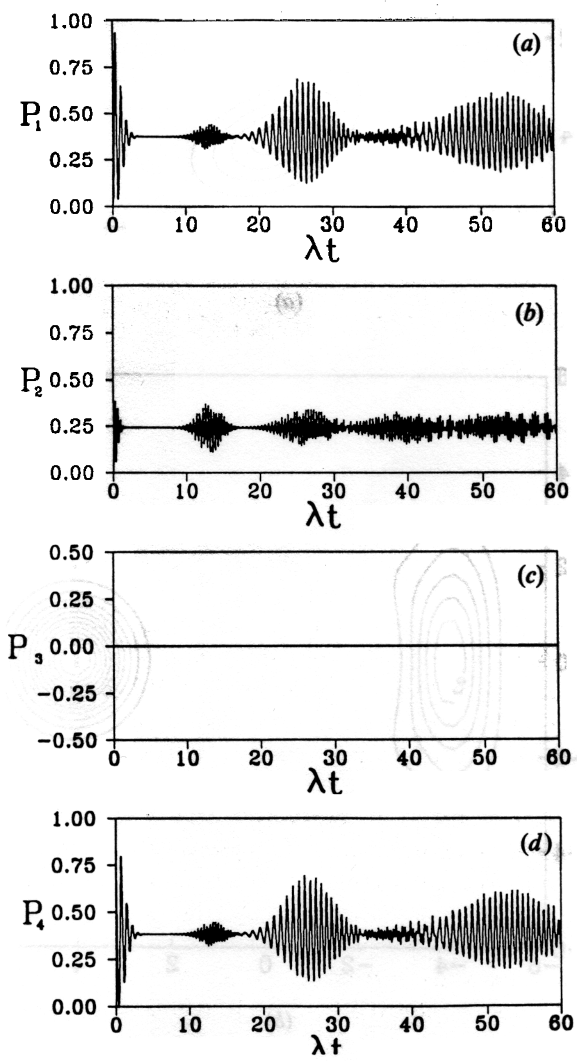
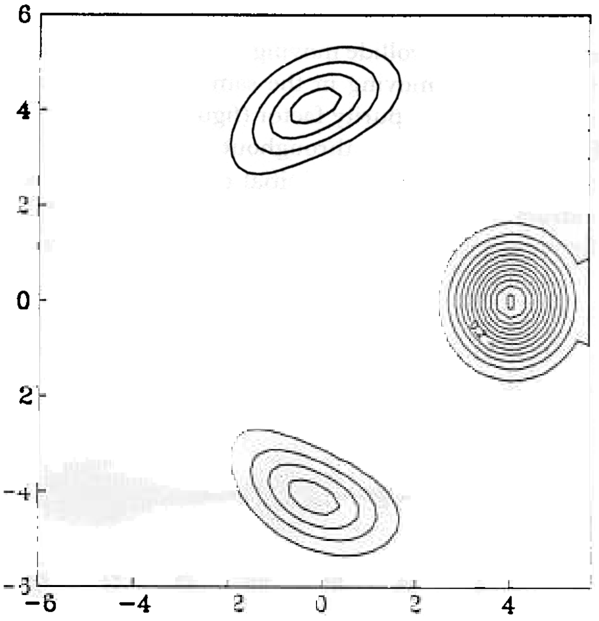
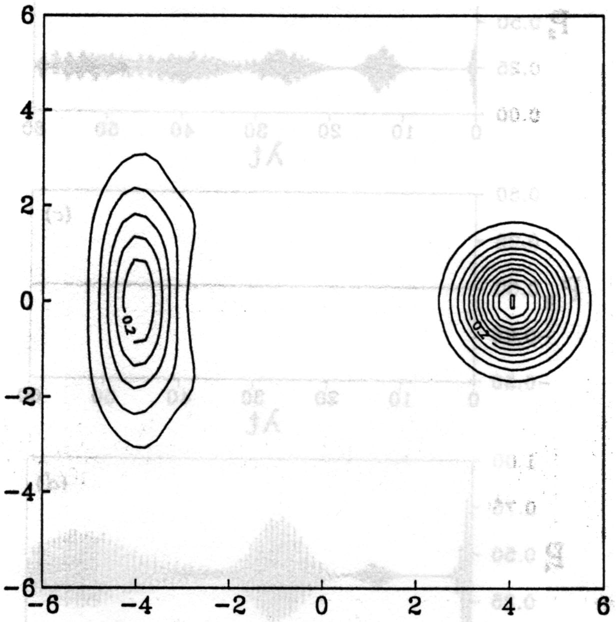


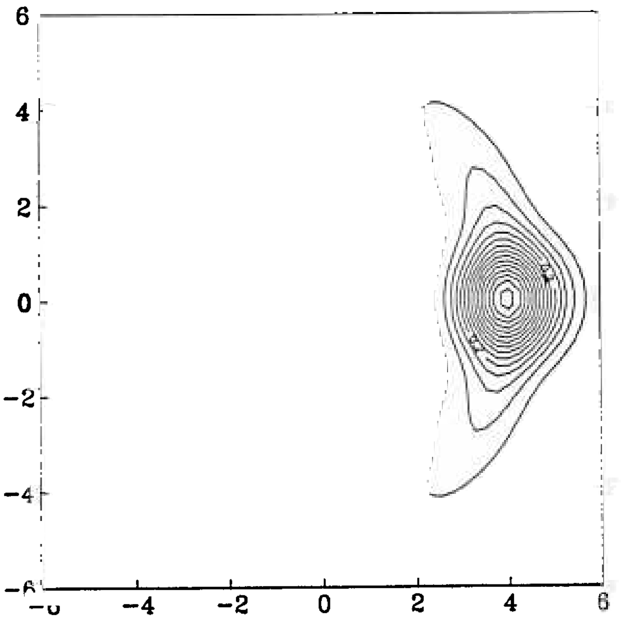
Figure 8. As figure 4, but for $kR=0$.



(*a*)



(*b*)



(c)

Figure 9. Contour plot of the field reduced density matrix Q function quasiprobability for an interatomic distance $kR=0$ with $\bar{n}=16$ for different moments of time: (a) $\lambda t=6{\cdot}25$; (b) $\lambda t=12{\cdot}5$; (c) $\lambda t=25$. Both atoms are initially excited. The figure shows the existence of a stationary blob in phase-space.

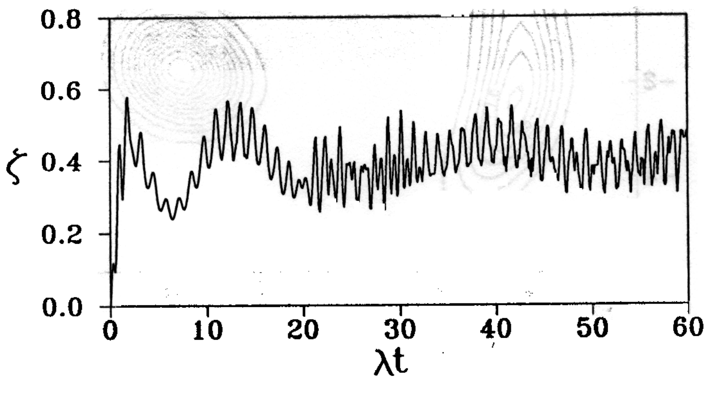
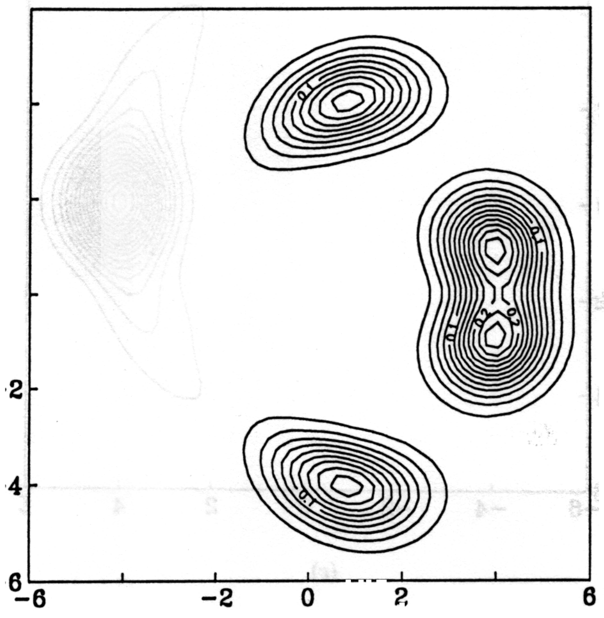
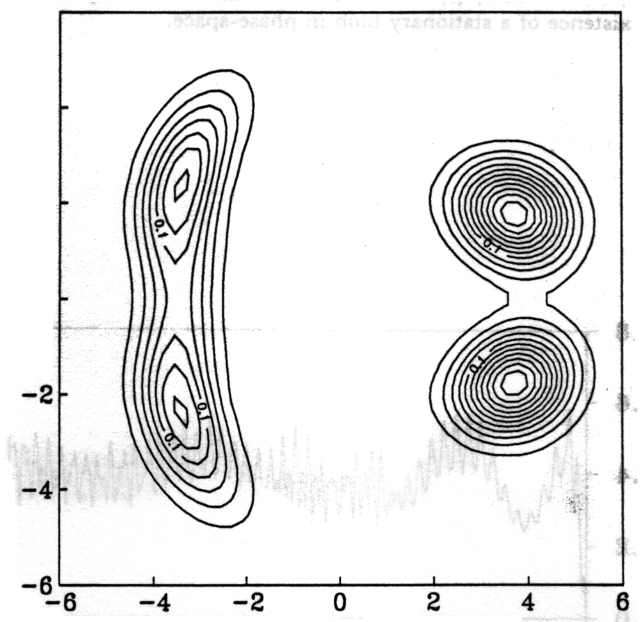


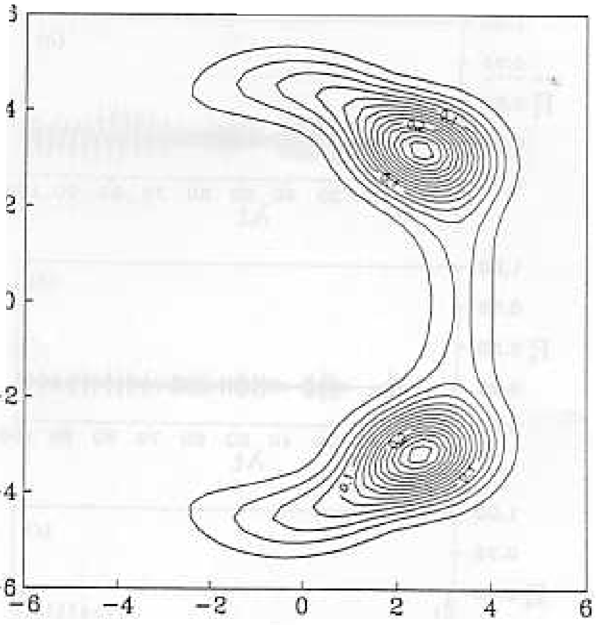
Figure 10. Field purity ξ as a function of λt; $kR=0$, $\bar{n}=16$. Both atoms are initially excited.



(*a*)



(*b*)



(c)

Figure 11. As figure 5, but for an interatomic distance $kR=\pi/4$.

$$b_1(t)=\mathrm{i}\exp(-\mathrm{i}at)\,[2(n+2)^{1/2}(1+\cos kR)(\chi_1^2-\chi_2^2)]^{-1}$$
$$\times\{\chi_2[2(n+2)(1+\cos kR)^2-\chi_1^2]\sin\mu_2 t-\chi_1[2(n+2)(1+\cos kR)^2$$
$$-\chi_2^2]\sin\mu_1 t\}, \qquad (27\,a)$$

$$b_2(t)=\exp(-\mathrm{i}at)[8(n+2)\cos kR(\chi_1^2-\chi_2^2)]^{-1}$$
$$\times\{[2(n+2)(1+\cos kR)^2-\chi_1^2][2(n+2)(1-\cos kR)^2-\chi_2^2]\cos\mu_2(t)$$
$$-[2(n+2)(1+\cos kR)^2-\chi_2^2][2(n+2)(1-\cos kR)^2-\chi_1^2]\cos\mu_1(t)\},$$

$$b_3(t)=\exp(-\mathrm{i}at)(1-\cos kR)[8(n+2)\cos kR(1+\cos kR)(\chi_1^2-\chi_2^2)]^{-1}$$
$$\times[2(n+2)(1+\cos kR)^2-\chi_1^2][2(n+2)(1+\cos kR)^2-\chi_2^2]$$
$$\times[\cos\mu_1(t)-\cos\mu_2(t)],$$

and

$$b_4(t)=\mathrm{i}\exp(-\mathrm{i}at)(n+1)^{-1/2}(1-\cos kR)[8(n+2)\cos kR(1+\cos kR)$$
$$\times(\chi_1^2-\chi_2^2)]^{-1}[2(n+2)(1+\cos kR)^2-\chi_1^2][2(n+2)(1+\cos kR)^2-\chi_2^2]$$
$$\times(\chi_1\sin\mu_1 t-\chi_2\sin\mu_2 t). \qquad (27\,d)$$

With these initial conditions for the case $kR=0$ the stationary blob will be absent even for the case of a three-level system. Moreover, as we can see from figure 15, even for an arbitrary distance we have two blobs rotating in phase space with a speed depending on kR. This speed will decrease as kR approaches π, for this case we have one stationary blob (in this case the Q function does not evolve at all

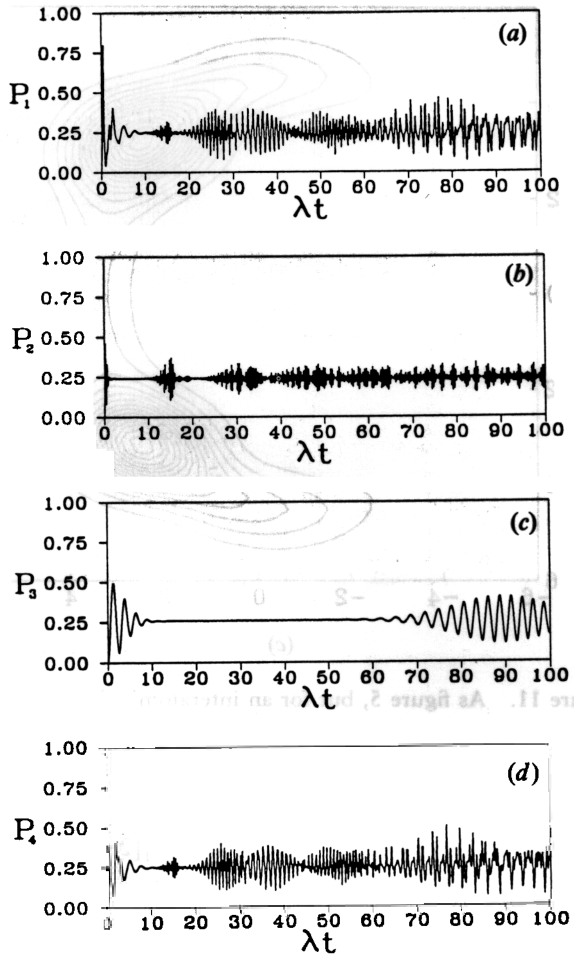


Figure 12. As figure 4, but for $kR=\pi/4$.

because the initially excited symmetric state decouples from the evolution). It can be seen from the plots for the purity factor (figure 16) that for these initial conditions we will have cat states for any distance (i.e. even for a four-level system) at the half revival time. When kR is close to π, we have an almost pure state (the purity parameter is very close to zero). Knight and Shore [5] noted a similar dependence of the purity on the initial conditions for a three-level atom. They noted that in general cat states were not produced if the three-level atom started either in its fully excited or fully de-excited state, but they were if the atom was initially prepared in its intermediate state provided the transition moments to upper and lower states were equal. This was attributed to a balancing or interference between competing transitions. In our four-level description of the two-atom states very similar interference can occur provided the initial condition is carefully chosen. Starting in the symmetric state, we see an identical sequence of coupling strengths for transitions round to the antisymmetric state via either the fully excited or fully de-excited intermediate states. These pathways can then interfere

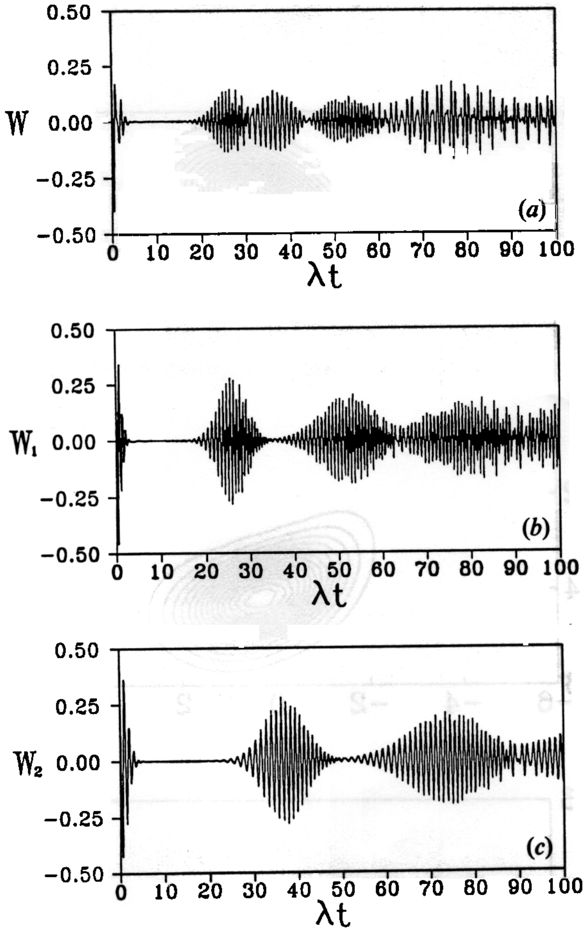


Figure 13. As figure 3, but for an interatomic distance $kR=\pi/4$.

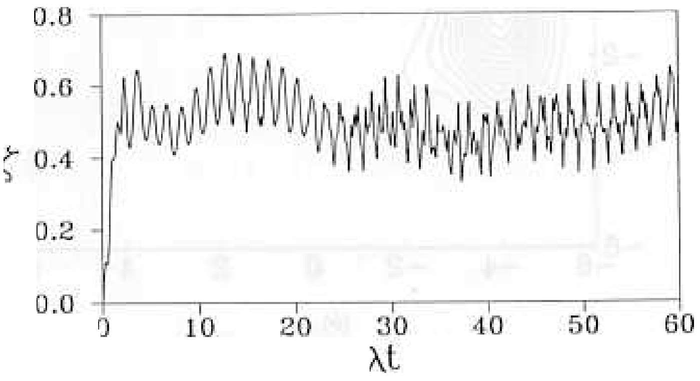
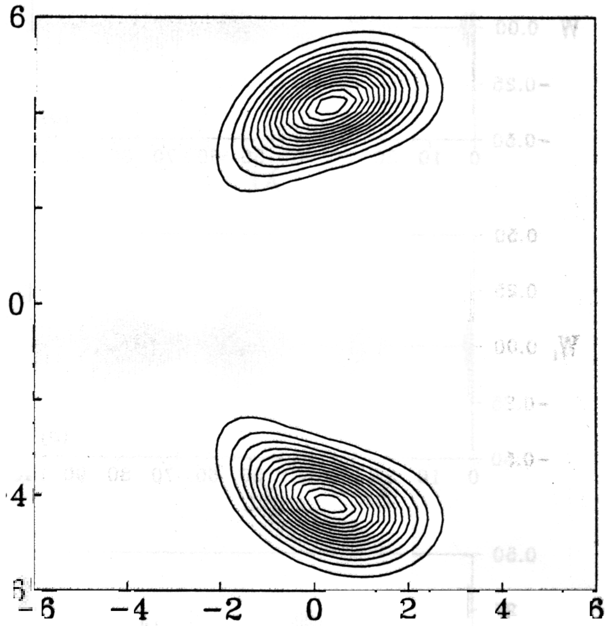
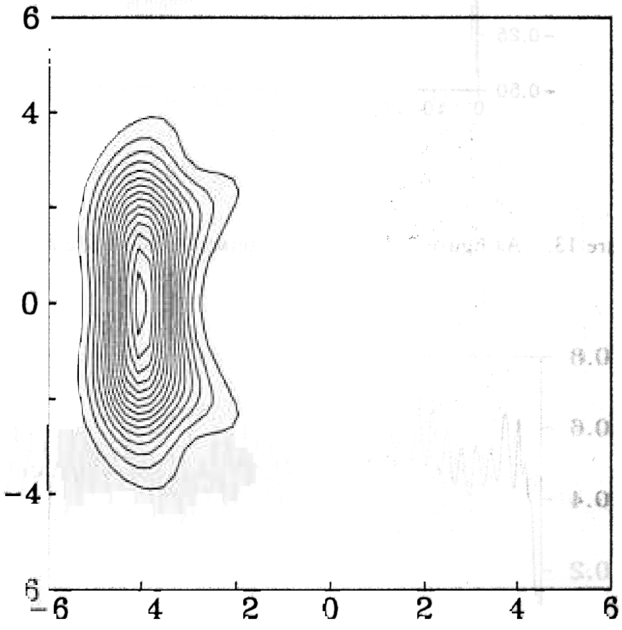


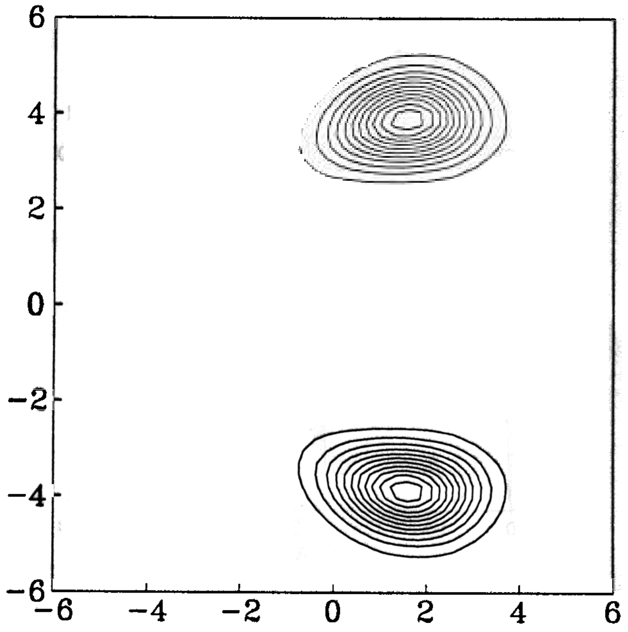
Figure 14. As figure 6, but for an interatomic distance $kR=\pi/4$.



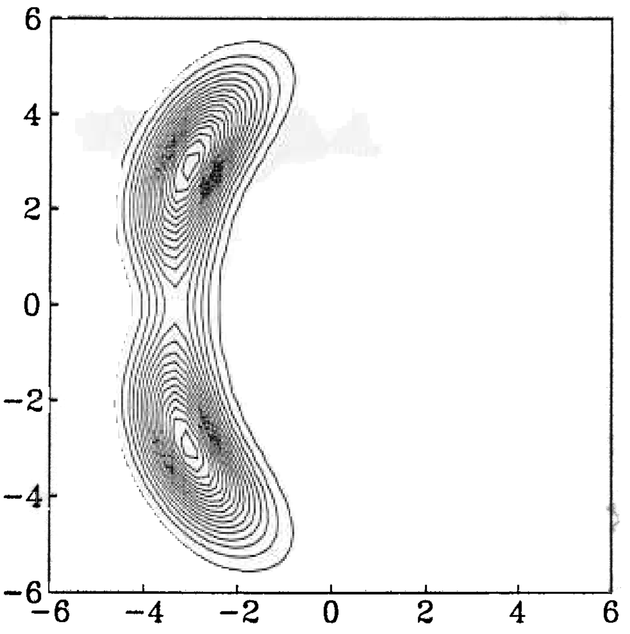
(a)



(b)



(*c*)



(*d*)

Figure 15. Contour plot of the field reduced density matrix Q function quasiprobability for the initially excited symmetric state and $\bar{n}=16$: (*a*) $kR=0$, $\lambda t=6{\cdot}25$, (*b*) $kR=0$, $\lambda t=12{\cdot}5$, (*c*) $kR=(3/10)\pi$, $\lambda t=6{\cdot}25$, (*d*) $kR=(3/10)\pi$, $\lambda t=12{\cdot}5$.

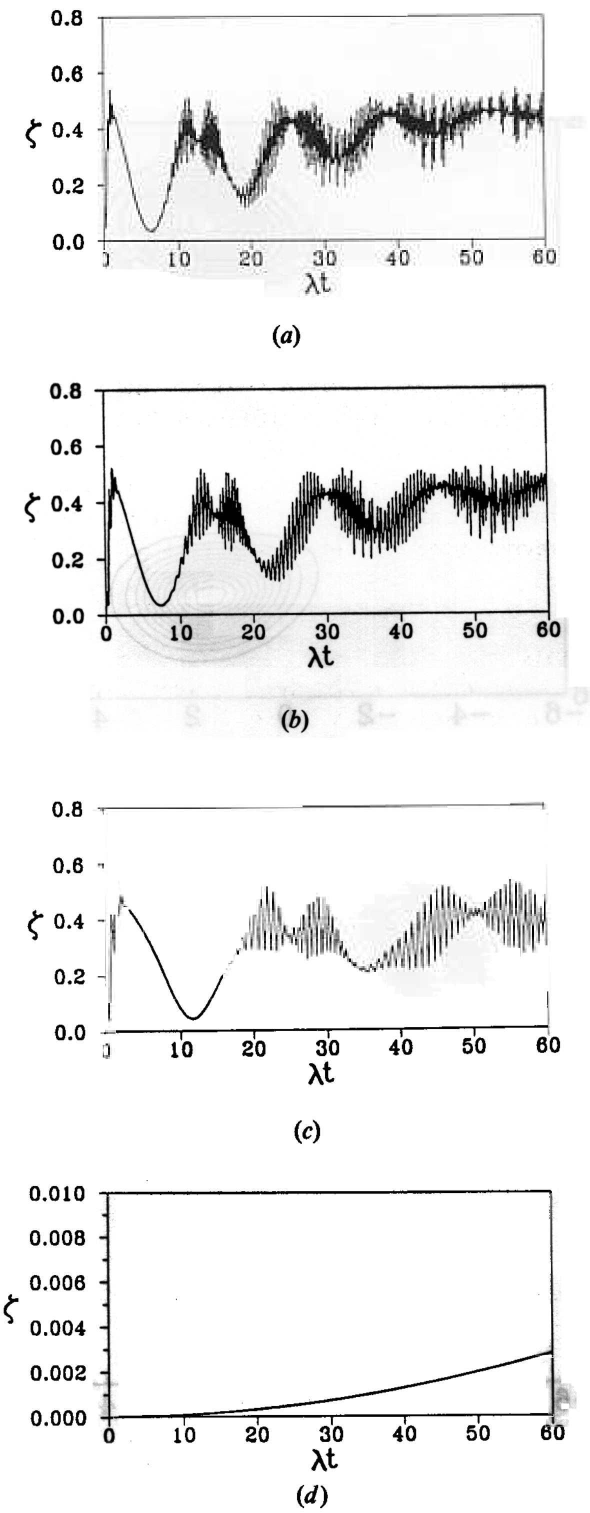


Figure 16. Field purity ζ as a function of λt for $\bar{n}=16$ starting in the symmetric atomic state as in figure 15 and different kR: (*a*) $kR=0$, (*b*) $kR=\pi/4$, (*c*) $kR=\pi/2$, (*d*) $kR \cong \pi$.

and it seems that again the interference is responsible for preserving the purity of the field.

5. Conclusions

When two two-level atoms interact with the cavity field mode, the problem of entanglement of field–atomic states becomes much more complicated than in the conventional one atom JCM. When initially both atoms are excited, only for the case $kR=\pi/2$ (when we have effectively a single two-level atom coupled to the field) are atomic and field states asymptotically disentangled at the half-revival time. This disentanglement is manifested by a value of a purity factor $\xi(T_R/2)\cong 0$ and for this case we have two rotating blobs in phase space. For any other distance (satisfying the condition $0\leqslant kR\leqslant\pi$) we have three (one stationary and two rotating) or four (all rotating) blobs in phase space and the purity factor is never close to zero so the Schrödinger cat state at the half revival time is spoilt. However, if we change the initial conditions we can generate two rotating blobs (as was the case when the symmetrical state $|\varphi_2\rangle$ was initially excited) the cat state will be restored and for the half revival time the atomic and field states will be asymptotically disentangled.

Acknowledgments

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