8

SISTEMAS ALGEBRAICOS: MATRICES

8 - 1

RESUMEN DE LA TEORÍA MATRICIAL

SISTEMAS LINEALES

DE ECUACIONES ALGEBRAICAS LINEALES

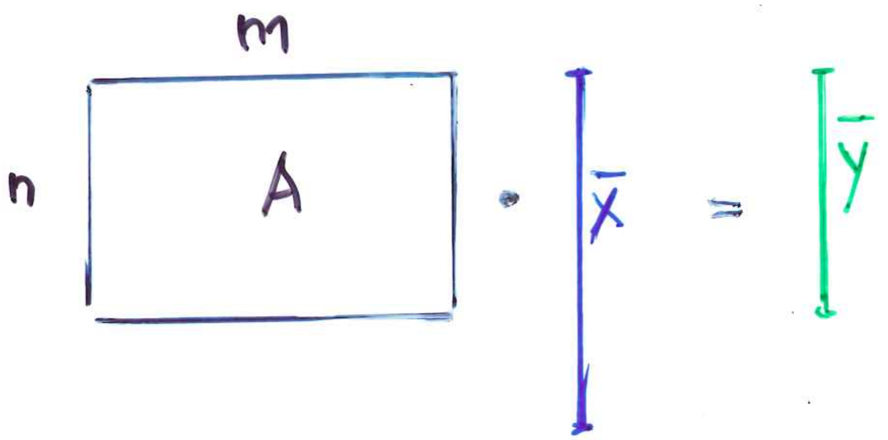
n ECUACIONES

$\bar{Y}$ VECTOR DE RESULTADOS DATOS
DE DIMENSION n

m INCOGNITAS

$\bar{X}$ VECTOR INCOGNITAS
DE DIMENSION m

$$A \cdot \bar{X} = \bar{Y}$$



MATRIZ A

ELEMENTOS A_{JK}

EN PRINCIPIO NOS OCUPAREMOS AQUÍ
DE MATRICES CUYOS ELEMENTOS A_{JK}
SON TODOS REALES

MATRIZ TRANSPUESTA $\tilde{A}$ $(m \times n)$

$$\tilde{A}_{JK} = A_{KJ}$$

CAMBIO DE FILAS POR COLUMNAS

MATRIZ CUADRADA $n = m$

M. UNIDAD

$$I = \begin{bmatrix} 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & 1 \end{bmatrix}$$

M. SIMETRICA $\tilde{A} = A$

M. ORTOGONAL matriz U definida por la propiedad $\tilde{U}U = U\tilde{U} = I$

M. INVERSA DE A ES A^{-1} TAL QUE

$$AA^{-1} = A^{-1}A = I$$

$$\widetilde{A \cdot B} = \tilde{B} \cdot \tilde{A} \qquad \widetilde{(A \cdot B \cdot C)} = \tilde{C} \cdot \tilde{B} \cdot \tilde{A}$$

$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1} \qquad (A \cdot B \cdot C)^{-1} = C^{-1} \cdot B^{-1} \cdot A^{-1}$$

DESCOMPOSICION CANONICA

valores propios λ

vectores propios $\bar{x}$

$$A \cdot \bar{x} = \lambda \bar{x}$$

M. CANONICA

$$\Lambda = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

EJES PRINCIPALES $\bar{u}_k$ DE A

VECTORES COLUMNA $\bar{u}_k$ DE LA MATRIZ U QUE SATISFACE LA ECUACION MATRICIAL

$$A \cdot U = U \cdot \Lambda$$

EJES PRINCIPALES $\bar{V}_J$ DE $\tilde{A}$

(ADJUNTOS DE LOS ANTERIORES)

VECTORES COLUMNA $\bar{V}_J$ DE LA MATRIZ V QUE SATISFACE LA ECUACIÓN MATRICIAL

$$\tilde{A} \cdot V = V \Lambda$$

CADA UNO DE LOS VECTORES $\bar{u}_k$ ES ORTOGONAL A CADA UNO DE LOS VECTORES $\bar{V}_J$

SE PUEDEN NORMALIZAR Y

TAL QUE

$$\tilde{V}\, U = \tilde{U}\, V = I$$

$$\Rightarrow \quad \tilde{V} = U^{-1} \qquad V = U^{-1}$$

$$U = \tilde{V}^{-1} \qquad \tilde{U} = V^{-1}$$

CADA MATRIZ ES LA INVERSA TRANSPUESTA DE LA OTRA

CON LO CUAL

$$A = U\, \Lambda\, \tilde{V} = U\, \Lambda\, U^{-1}$$

$$\tilde{A} = V\, \Lambda\, \tilde{U} = V\, \Lambda\, V^{-1}$$

JORDAN

VALORES PROPIOS DE LA INVERSA

SI $A\bar{x} = \lambda \bar{x}$ $\qquad A^{-1} A\, \bar{x} = \lambda A^{-1} \bar{x}$

$$A^{-1}\bar{x} = \frac{1}{\lambda}\bar{x}$$

VECTORES PROPIOS : LOS MISMOS

VALORES PROPIOS : LOS INVERSOS

VALORES PROPIOS DE LA TRANSPUESTA

LOS MISMOS POR CONSTRUCCION

LOS VALORES PROPIOS DE UNA MATRIZ CUALQUIERA, PUEDEN SER COMPLEJOS (AUNQUE LA MATRIZ SEA REAL)

LOS VALORES PROPIOS DE UNA MATRIZ SIMETRICA, SON SIEMPRE REALES
LOS VECTORES PROPIOS SON ORTOGONALES

MATRIZ SIMETRICA PARTICULAR

QUE ENCONTRAREMOS FRECUENTEMENTE

PRODUCTO $\tilde{A} . A$

PARA ESTA MATRIZ SIMETRICA PARTICULAR LOS VALORES PROPIOS SON SIEMPRE REALES Y POSITIVOS

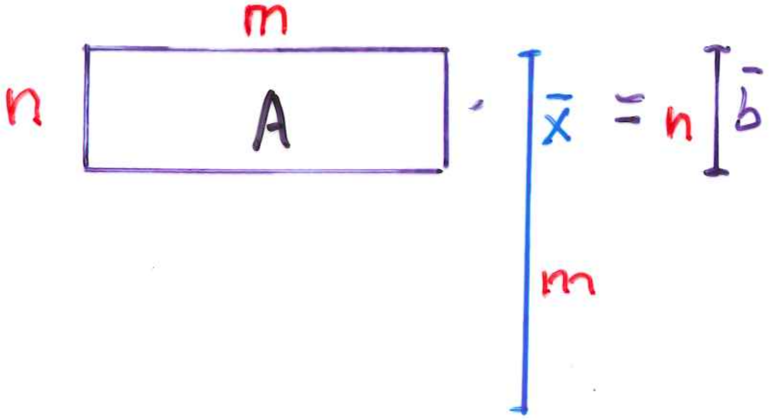
UNA MATRIZ SIMETRICA REAL SE PUEDE DESCOMPONER COMO PRODUCTO DE DOS MATRICES TRIANGULARES

$$A = T_L \tilde{T}_L$$

8-2.

DESCOMPOSICION ESPECTRAL SINGULAR

SISTEMA LINEAL ARBITRARIO $n \times m$

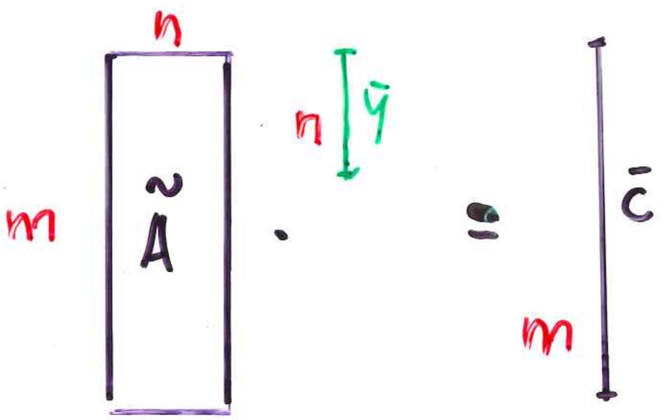


$$A.\bar{x} = \bar{b}$$

NO TIENE NINGUN SIGNIFICADO EL PROBLEMA DE SUS VALORES PROPIOS

UN PROBLEMA SIMILAR, ADJUNTO DEL ANTERIOR, CON LA MATRIZ TRANSPUESTA

$\tilde{A}$ $m \times n$



$$\tilde{A}.\bar{y} = \bar{c}$$

JUNTEMOS LOS DOS PROBLEMAS EN UN SÓLO SISTEMA :

$$\begin{bmatrix} 0 & A \\ \tilde{A} & 0 \end{bmatrix} \cdot \begin{bmatrix} \bar{y} \\ \bar{x} \end{bmatrix} = \begin{bmatrix} \bar{b} \\ \bar{c} \end{bmatrix}$$

DIMENSION $(n+m) \times (n+m)$ MATRIZ S

NO SE HA ALTERADO PARA NADA EL PROBLEMA INICIAL DE RESOLVER LOS DOS SISTEMAS INDEPENDIETES

ESTÁN FORMULADOS COMO UN TODO PERO NO SE HAN MEZCLADO

PERO AHORA LA MATRIZ S

ES SIMÉTRICA

TIENE AUTOVALORES REALES

$$S \cdot \bar{w} = \lambda \, \bar{w}$$

ECUACION DE LOS AUTOVALORES DE S

DADA LA FORMA DE S

SI DESCOMPONEMOS LAS $n+m$ COMPONENTES DE $\overline{W}$, EN DOS GRUPOS: DOS VECTORES $\overline{u}$ Y $\overline{v}$ DE DIMENSIONES n Y m RESPECTIV.

$$\overline{W} \equiv (\overline{u}, \overline{v})$$

LA ECUACIÓN ANTERIOR PARA S:

$$S \cdot \overline{W} = \lambda \overline{W}$$

SE PUEDE DESCOMPONER EN DOS

$$\begin{cases} A \cdot \overline{v} = \lambda \overline{u} & \overline{v} \text{ DIMENSIÓN } m \\ \tilde{A} \cdot \overline{u} = \lambda \overline{v} & \overline{u} \text{ DIMENSIÓN } n \end{cases}$$

LANCZOS LLAMA A ESTAS DOS ECUACIONES SHIFTED EIGENVALUE PROBLEM

EL MISMO AUTOVALOR λ.

DOS "VECTORES IMPROPIOS"
PORQUE LA DIMENSIÓN DEL ESPACIO DE ENTRADA NO TIENE POR QUE SER IGUAL A LA DIMENSIÓN DEL ESPACIO DE SOLIDA
SI LA MATRIZ NO ES CUADRADA

APLICACIÓN DE ESTA IDEA A OTROS OPERADORES LINEALES.

COMO S ES SIMETRICA

VECTORES PROPIOS $\bar{W}$ ORTOGONALES

$\tilde{\bar{W}} \cdot \bar{W} = I \Rightarrow$

$$\tilde{\bar{u}}_i \bar{u}_k + \tilde{\bar{V}}_i \cdot \bar{V}_k = 0 \qquad i \neq k$$

$\forall \lambda_i, \lambda_k$

PERO SI SE CUMPLE EL ANTERIOR SHIFTED EIGENVALUE PROBLEM

TAMBIEN SE CUMPLE EL

$$A \cdot \bar{V} = -\lambda(-\bar{u})$$

$$\tilde{A} \cdot (-\bar{u}) = -\lambda \bar{V}$$

CAMBIANDO DE SIGNO λ Y $\bar{u}$

COMBINANDO AHORA LAS SOLUCIONES PARA λ_i Y $-\lambda_k$

$$\tilde{\bar{u}}_i \bar{u}_k - \tilde{\bar{V}}_i \bar{V}_k = 0 \qquad i \neq k$$

$$\Rightarrow \begin{cases} \tilde{\bar{u}}_i \cdot \bar{u}_k = 0 \\ \tilde{\bar{V}}_i \cdot \bar{V}_k = 0 \end{cases}$$

CADA VECTOR EN SU ESPACIO ES ORTOGONAL

A PARTIR DE LAS ECUACIONES

$$\begin{cases} A \cdot \bar{V} = \lambda \, \bar{u} \\ \tilde{A} \cdot \bar{u} = \lambda \, \bar{V} \end{cases}$$

$$\tilde{A} \cdot A \cdot \bar{V} = \lambda \, \tilde{A} \, \bar{u} = \lambda^2 \, \bar{V}$$

$\tilde{A} \cdot A$ MATRIZ CUADRADA SIMETRICA DE DIMENSION $m \times m$

λ^2 SU AUTOVALOR POSITIVO

$\bar{V}$ VECTOR PROPIO DIMENSIÓN m

$$A \cdot \tilde{A} \cdot \bar{u} = \lambda \, A \cdot \bar{V} = \lambda^2 \, \bar{u}$$

$A \cdot \tilde{A}$ MATRIZ CUADRADA SIMETRICA DE DIMENSION $n \times n$

λ^2 SU AUTOVALOR POSITIVO

$\bar{u}$ VECTOR PROPIO DIMENSIÓN n

8.3

SISTEMAS ALGEBRAICOS

- CLASIFICACIÓN
- COMPATIBILIDAD
- MÍNIMOS CUADRADOS

Classification of linear systems

One of the peculiarities of linear systems is that our naive notions concerning enumeration of equations and unknowns fail to hold. On the surface we would think that n equations suffice for the determination of n unknowns. We would also assume that having less equations than unknowns our system will have an infinity of solutions. Both notions can easily be disproved. The following system of three equations with three unknowns

$$x + y + z = 1$$
$$2x + 2y + 2z = 2$$
$$x - y + z = 3$$

is clearly unable to determine the 3 unknowns x, y, z since in fact we have only *two* equations, the second equation being a mere repetition of the first. Moreover, the following system of two equations for three unknowns

$$x + y + z = 1$$
$$2x + 2y + 2z = 3$$

is far from having an infinity of solutions. It has no solution at all since the second equation contradicts the first one. This can obviously happen with *any* number of unknowns, and thus an arbitrarily small number of equations (beyond 1) with an arbitrarily large number of unknowns may be self-contradictory.

The only thing we can be sure of is that a linear system can have no unique solution if the number of equations is less than the number of unknowns.

The number n of equations and the number m of unknowns are here no longer matched but generally

$n < m$ (under-determined)
$n = m$ (even-determined)
$n > m$ (over-determined)

and accordingly we speak of under-determined, even-determined and over-determined linear systems.

COMPATIBILIDAD

In order to develop the proper critical faculty for realistic appraisal of strongly skew-angular systems, we have first to develop the mathematical theory of the compatibility of linear systems and then properly modify it in order to apply it to the question of the physical feasibility of a given set of linear equations.

The mathematical compatibility problem of linear systems arises from the following consideration. If a problem contains n unknowns, our first thought is to get n equations for the determination of these unknowns. If the number of equations is less than n, we know in advance that the given information will not suffice for unique determination of all the unknowns. If the number of equations is more than n, we have an abundance of information which will generally lead to contradictions. Hence we discard underdetermined systems because they contain too little information and cannot lead to a unique solution of the problem, and we discard overdetermined systems because they contain too much information and serve no useful purpose.

strongly skew-angular:
mal condicionado

It is important to realize, however, that the compatibility of a given set of equations bears no relation to the underdetermined or overdetermined or evendetermined ("balanced") nature of the problem. The following two equations are underdetermined, since two equations are given for five unknowns.

$$x_1 + x_2 + x_3 + x_4 + x_5 = 3 \tag{2-24.3}$$

$$2x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 = 8$$

But these two equations are self-contradictory and cannot be solved for any values of the five unknowns. On the other hand, the following five equations are given for only two unknowns and thus the system is overdetermined.

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + 3x_2 &= -1 \\ 3x_1 + 2x_2 &= 1 \\ x_1 - x_2 &= 2 \\ 3x_1 + 5x_2 &= -2 \end{aligned} \tag{2-24.4}$$

These five equations are not contradictory, but have the solution $x_1 = 1$, $x_2 = -1$.

The question arises whether there is a systematic way of deciding the compatible or incompatible nature of a given set of equations. Such a systematic method exists indeed and can be described as follows. Let us consider the linear equation

$$Ax = b \tag{2-24.5}$$

and let us augment it by the adjoint equation

$$\tilde{A}y = c \qquad \forall \bar{c} \tag{2-24.6}$$

We form the scalar product of the first equation with y, the second equation with x:

$$y \cdot Ax = y \cdot b, \qquad x \cdot \tilde{A}y = x \cdot c \tag{2-24.7}$$

Now by the fundamental transposition rule (8.23) we have

$$y \cdot Ax = x \cdot \tilde{A}y \tag{2-24.8}$$

This shows that the left sides of equations (7) are equal for any choice of x and y. Hence the right sides must also be equal and we obtain

$$y \cdot b - x \cdot c = 0 \tag{2-24.9}$$

This equation is of no particular use, however, in deciding the compatibility of the system $Ax = b$, since it demands the knowledge of x (i.e., the solution of the given system), while our aim is to decide whether the system is solvable at all. In one particular case, however, the vector x will drop out from our relation; viz., if c happens to be zero. Then we get

$$y \cdot b = 0 \tag{2-24.10}$$

where y is the solution of the equation

$$\tilde{A}y = 0 \tag{2-24.11}$$

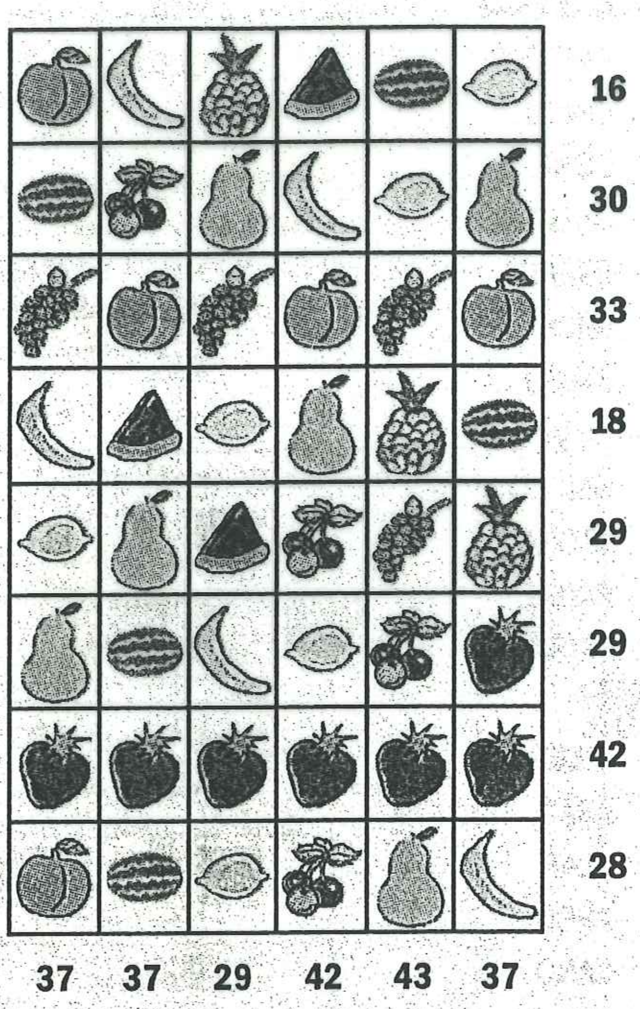
One can show that this condition is not only necessary but also *sufficient*. We thus obtain the following general principle which answers all compatibility problems of linear systems: "The right side of a given set of linear equations has to be orthogonal to any solution of the adjoint homogeneous equation."

This general principle operates in a given case in a variety of ways. First, it is possible that the adjoint homogeneous equation $\tilde{A}y = 0$ has *no solution* outside of the identical vanishing of y. In that case we do not get any compatibility condition, which means that the given set (5) is compatible with *any* given right side. Second, it is possible that the adjoint homogeneous equation $\tilde{A}y = 0$ has one and *only one* solution (not counting the trivial solution $y = 0$ and not counting the freedom of an arbitrary factor by which y can be multiplied, because of the homogeneity of the equation). In this case the given right side has to satisfy one compatibility condition by being orthogonal to the homogeneous adjoint solution. Third, it is possible that the adjoint homogeneous equation $\tilde{A}y = 0$ has a number of independent solutions. In that case the given right side has to be orthogonal to *every one* of these independent solutions.

The question of overdetermination or underdetermination or evendetermination does not enter specifically the application of this general principle except for the fact that in the case of an overdetermined system the adjoint set has *always* nontrivial solutions, and thus the given right side must always satisfy one or more conditions.

The decision that a certain system of equations is compatible does not necessarily imply that the solution of the system is *unique*. A compatible system of equations may have one or more solutions. In our first example, we had an underdetermined incompatible system which had no solutions. In our second example we had an underdetermined compatible system which has an infinity of solutions. In our third example we have an overdetermined compatible system which has a unique solution. Generally we can say that if a compatibility condition is satisfied, we can drop one of the equations as superfluous. An overdetermined system will thus turn to an evendetermined system. But it is also possible that it may turn into an underdetermined system.

SISTEMA LINEAL BIEN PLANTEADO



Sabiendo que se trata de sumas horizontales y verticales, averigüe qué dígito corresponde a cada fruta.

MANZANA	x_1
PLATANO	x_2
PIÑA	x_3
SANDIA	x_4
MELON	x_5
LIMON	x_6
CEREZA	x_7
PERA	x_8
UVAS	x_9
FRESA	x_{10}

$$A\bar{x} = \bar{b}$$

A	(14×10)	MATRIX	
$\bar{x}$	(10)	VECTOR	$m = 10$
$\bar{b}$	(14)	VECTOR	$n = 14$

OVER-DETERMINED $(n > m)$

ESPEREMOS QUE LAS ECUACIONES SEAN COMPATIBLES.

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 16$$

$$x_2 + x_5 + x_6 + x_7 + 2x_8 = 30$$

$$3x_1 + 3x_9 = 33$$

$$x_2 + x_3 + x_4 + x_5 + x_6 + x_8 = 18$$

$$x_3 + x_4 + x_6 + x_7 + x_8 + x_9 = 29$$

$$x_2 + x_5 + x_6 + x_7 + x_8 + x_{10} = 29$$

$$6x_{10} = 42$$

$$x_1 + x_2 + x_5 + x_6 + x_7 + x_8 = 28$$

$$2x_1 + x_2 + x_5 + x_6 + x_8 + x_9 + x_{10} = 37$$

$$x_1 + x_2 + x_4 + 2x_5 + x_7 + x_8 + x_{10} = 37$$

$$x_2 + x_3 + x_4 + 2x_6 + x_8 + x_9 + x_{10} = 29$$

$$x_1 + x_2 + x_4 + x_6 + 2x_7 + x_8 + x_{10} = 42$$

$$x_3 + x_5 + x_6 + x_7 + x_8 + 2x_9 + x_{10} = 43$$

$$x_1 + x_2 + x_3 + x_5 + x_6 + x_8 + 2x_{10} = 37$$

SISTEMA LINEAL BIEN PLANTEADO

TANTO EN LOS DATOS, COMO EN LOS COEFICIENTES : MATRIZ

APPLIED ANALYSIS

C. LANCZOS

The mere mathematical solution of a set of linear equations frequently blinds us to the dangers which arise in connection with large linear systems. The temptation is to use the large-scale computing facilities of the big electronic digital calculators for solution of extensive linear systems, without realizing that the exact mathematical solution obtained in this manner may have no physical significance whatever. The question concerning the physical significance of a mathematically correct solution has to be raised and the problem of "noise" has to be discussed. The "noise" here in question does not refer to the "arithmetical noise" caused by the rounding errors of our calculations, but to the "physical noise" caused by the inexactitude of our measurements.

EJEMPLO →

While in this simple example we can follow each detail of the situation and demonstrate explicitly the unsatisfactory nature of nearly singular systems, we frequently accept the results deduced from strongly skew-angular systems without realizing that in view of the physical noise of the problem the mathematical solution may have little relation to the true values of the quantities that our solution is supposed to yield.

MAL CONDICIONADOS

COMPATIBILIDAD CON RUIDO

→ EJEMPLO: COMPATIBILIDAD EN PRESENCIA DE RUIDO

$$\begin{cases} x+y = 2.00001 \\ x+1.00001\,y = 2.00002 \end{cases}$$

SOLUCIÓN $\quad x = 1.00001$
$\quad y = 1$

PERO EL SEGUNDO MIEMBRO ES EL RESULTADO DE UNA MEDIDA FISICA Y PUEDE NO TENER LA PRECISIÓN NECESARÌA

LAS ECUACIONES PUEDEN ENTONCES, NO SER COMPATIBLES.

REESCRIBIMOS:

$$\begin{cases} (x+y) = 2.00001 \\ 1.000005\,(x+y) - 0.000005\,(x-y) = 2.00002 \end{cases}$$

ES FACIL DE ENCONTRAR $(x+y)$ PERO (SALVO PRECISIÓN MUY GRANDE) NO PODEMOS OBTENER $(x-y)$

SOBRE-DETERMINACIÓN

MINIMOS CUADRADOS

MUCHAS ECUACIONES ⇒ MUCHO RUIDO

is the "noise" which interferes with the accuracy of our measurements and distorts the true course of events. Since noise is of a random nature, the distortion is not consistent but occurs once in one, once in the other direction. This is *one* danger encountered in large-scale recordings of physical events. The *other* danger is that the information we have at our disposal is *insufficient* for actual determination of all the unknowns of the problem.

In analogy to this situation it can happen (and it frequently does happen) that the statements of our system of equations are insufficient for complete determination of all the unknowns of our problem. We count the number of equations and find that we have just as many equations as unknowns. Hence we think that our system is balanced and allows a unique solution. Yet it can happen that certain equations merely repeat in different words the statements made before, without adding anything essentially new to the previous statements. In this case our system is underdetermined and not in the position to yield a complete solution of our problem.

The two difficulties are interconnected. If the left sides of the equations are interdependent, the right sides have to satisfy certain compatibility conditions. But these conditions may not be satisfied in view of the noise of the measurements, which renders our equations incompatible in the strict mathematical sense. Coupled with this incompatibility is the fact that our equations are not sufficient for determination of all the unknowns of our problem, since in the case of compatibility we could drop a certain number of equations as superfluous, in which case we have not enough equations for the complete solution.

PRECAUCIONES: generalmente puede faltar información, aunque creamos que sobra

The problems of underdetermination and overdetermination are thus interlocked. Our system is in fact underdetermined because of absence of certain linear combinations of the unknowns, which makes it impossible to obtain *all* the unknowns of the system. A corresponding number of equations becomes superfluous and could be dropped. Hence an n by n set of equations which omits m linear combinations of the unknowns represents in reality a set of $n - m$ equations in $n - m$ unknowns, and is thus overdetermined by m equations.

One of the two difficulties is solvable; viz., the problem of overdetermination. The ingenious method of least squares makes it possible to adjust an arbitrarily overdetermined and incompatible set of equations. In fact, we make an asset out of a liability and try to overdetermine a set of equations as much as possible by making an arbitrary number of surplus observations beyond the minimum number demanded by the number of unknowns. We now have a linear set of equations, characterized by

$$Ax = b \tag{2-25.1}$$

where A is not a square matrix, but a matrix which has many more rows than columns. In view of the errors of our measurements, the equations become mathematically incompatible. This means that we cannot make all the components of the residual vector $Ax - b$ equal to zero.

But now we can ask for the "best" solution which is still available under the given circumstances. For this purpose we form the residual vector

$$Ax - b = r \tag{2-25.2}$$

and now we take the square of the length of the residual vector and determine x by the condition that r^2 shall become a minimum. The problem of minimizing $(Ax - b)^2$ has always a definite solution, no matter how compatible or incompatible the given system is. If the given system is compatible, the least square solution x, if substituted in $Ax - b$, will automatically give 0. If the given system is incompatible, the residual vector $Ax - b$ will not give zero but the solution of smallest error in the sense that the sum of squares of the residuals will be smaller than that for any other choice of x.

The least-square solution thus completely dispenses with the investigation of the compatibility of a given system since we are reconciled to the fact that we do not get the exact solution of our problem, but the best solution possible under the given circumstances. If the given system is compatible, the residual vector of the least-square solution becomes automatically zero, thus proving that the system is compatible.

The least-square solution of the equation $Ax = b$ becomes

$$\tilde{A}Ax = \tilde{A}b \tag{2-25.3}$$

The remarkable fact about this equation is that it always gives an evendetermined system of just as many equations as we have unknowns, no matter how strongly overdetermined the original system has been. We see at once from the figure that the product of an m row, n column matrix multiplied by an n row, m column matrix gives an m by m square matrix, and thus in the final set the number of equations balances the number of unknowns, no matter how many equations the original system contained. Moreover, the matrix $\tilde{A}A$ is always a symmetric matrix,

$$\widetilde{(\tilde{A}A)} = \tilde{A}A \tag{2-25.4}$$

and its eigenvalues are not only real but *positive* or in the limit zero.

On the other hand, even the least-square formulation of an underdetermined system cannot help us in getting a unique solution. If certain combinations of the unknowns are missing in our equations, there is no magic by which they could be conjured up. An underdetermined system remains thus underdetermined, even in the least-square formulation. However, an incompatible underdetermined system is transformed into a compatible underdetermined system.

MINIMOS CUADRADOS

$$\underline{\underline{K}}\,\bar{f} - \bar{g} = \bar{R}$$

MATRIZ $\underline{\underline{K}}$ $(M \times N)$ $M > N$

MAS FILAS QUE COLUMNAS

MAS ECUACIONES QUE INCOGNITAS

$\bar{R}$ VECTOR RESIDUOS

QUE HAY QUE MINIMIZAR

MINIMIZAREMOS EL CUADRADO DE $\bar{R}$

ES DECIR $\underline{R}\cdot\bar{R}$

(PRODUCTO ESCALAR)

MINIMIZAR

$$(\underline{\underline{K}}\,\bar{f} - \bar{g}) \cdot (\underline{\underline{K}}\,\bar{f} - \bar{g})$$

O SEA

$$(\underline{f}\,\underline{\underline{K}}^T - \underline{g}) \cdot (\underline{\underline{K}}\,\bar{f} - \bar{g}) =$$

$$= \underline{f}\,\underline{\underline{K}}^T\underline{\underline{K}}\,\bar{f} - \underline{g}\,\underline{\underline{K}}\,\bar{f} - \underline{f}\,\underline{\underline{K}}^T\bar{g} + \underline{g}\,\bar{g}$$

MINIMIZAR

$$\underline{f}\,\underline{\underline{K}}^T\,\underline{\underline{K}}\,\bar{f} - \underline{g}\,\underline{\underline{K}}\,\bar{f} - \underline{f}\,\underline{\underline{K}}^T\,\bar{g} + \underline{g}\,\bar{g}$$

$$\underline{f} = (f_1, f_2, \ldots f_J, \ldots f_N) \qquad \bar{f} \equiv \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_J \\ \vdots \\ f_N \end{pmatrix}$$

DERIVANDO CON RESPECTO A f_J

$$(0, 0, \ldots 1, \ldots 0)\,\underline{\underline{K}}^T\,\underline{\underline{K}}\,\bar{f} + \underline{f}\,\underline{\underline{K}}^T\,\underline{\underline{K}} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} -$$

$$- \underline{g}\,\underline{\underline{K}} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} - (0, 0, \ldots 1 \ldots 0)\,\underline{\underline{K}}^T \cdot \bar{g} = 0$$

SE CUMPLE SI

$$\underline{\underline{K}}^T\,\underline{\underline{K}}\,\bar{f} = \underline{\underline{K}}^T\,\bar{g}$$

EN UN SISTEMA CON MAS ECUACIONES QUE INCOGNITAS

EN PRINCIPIO:

SOBRE-DETERMINADO

PUEDEN FALTAR CIERTAS COMBINACIONES DE LAS INCOGNITAS

Y ENTONCES ES IMPOSIBLE ENCONTRAR ESTAS

AUNQUE SOBREN ECUACIONES Y APLIQUEMOS EL MÉTODO DE MINIMOS CUADRADOS

8-4

SOLUCIÓN DE SISTEMAS

INVERSIÓN MATRICIAL

- PERTURBACIONES
- ESTABILIDAD

INVERSION DE UNA MATRIZ

O RESOLUCIÓN DE UN SISTEMA LINEAL

The numerical aspects of inverting a matrix are distinctly different from the purely mathematical aspects. Our calculations are always of only limited accuracy. The rounding errors constantly accumulate, and considering the large number of operations involved in the inversion process, they may eventually obliterate the desired results. The Gaussian elimination scheme insures proper results if the effect of rounding errors can be neglected. But when inverting large matrices this is almost never the case. The numerical inversion of such matrices is of paramount interest, considering the fact that so many problems of contemporary physics and engineering lead to the solution of large linear systems. What procedure shall we follow under these circumstances?

The establishment of electronic digital computers started a new chapter in the history of numerical analysis. The extraordinary rapidity with which these machines perform the fundamental operations of arithmetic leads to a shift in our general philosophy of numerical operations. The emphasis is no longer on procedures which obtain a result in the smallest number of operations. More important is the viewpoint of *simple codibility*, together with the demand on high accuracy. In the problem of inverting a matrix we will be interested in a procedure which, in spite of the limited accuracy of arithmetical operations, cannot come to grief, no matter how extensive the matrix is to which it is applied.

In discussing the question of accuracy we must realize that under no circumstances can we expect *absolute* accuracy. Nor is the question of accuracy a matter of arbitrary decisions.

Rarely are the elements of the given matrix A absolute mathematical numbers. In most cases the elements a_{ik} of the matrix are obtained on the basis of some *measurements*, which are automatically of limited accuracy only. Let us now assume that as the result of some inversion process we have obtained an approximate inverse of A. This $\bar{A}^{-1}$ is certainly the exact inverse of a certain $\bar{A}$ which differs from A by the small amount αA_1:

$$\bar{A} = A + \alpha A_1$$

Let it be true that all the elements of αA_1 are smaller than the possible errors of a_{ik}. Under these circumstances we can describe $\bar{A}$ as being "numerically equivalent" to A, and the obtained $\bar{A}^{-1}$ can be accepted as the correct answer to our problem. It would be entirely superfluous to try to correct $\bar{A}^{-1}$ in order to arrive at the correct A^{-1}. The mathematically correct answer has no significance, in view of the limited accuracy of the given A.

In the case of a "mathematical matrix" A, the substitution of a "numerically equivalent" $\bar{A}$ loses, of course, all significance. But even then it will be advisable to replace the exact elements of A by elements limited to a definite number of decimal places, carry through the inversion process and, finally, if necessary, correct $\bar{A}^{-1}$ by a perturbation process

CORRECCION PERTURBATIVA DE LA SOLUCIÓN DE UN SISTEMA

$$A \cdot \bar{x} = \bar{b}$$

supongamos que hemos tenido una solución aproximada

$$\bar{x}_0$$

$$\bar{b} - A\,\bar{x}_0 = \bar{\varepsilon}_1 \quad \text{vector residual}$$

PLANTEAMOS

$$A \cdot \bar{x} = \bar{\varepsilon}_1$$

del cual tenemos una solución aproximada

$$\bar{x}_1$$

etc

$$\bar{x} = \bar{x}_0 + \bar{x}_1 + \ldots$$

CORRECCIÓN PERTURBATIVA DE LA INVERSION

A MATRIZ DADA

A^{-1} SU INVERSA TEORICA

$$A A^{-1} = A^{-1} A = I$$

CON UN METODO DE INVERSIÓN ENCONTRAMOS

$$A_a^{-1}$$

TAL QUE $$A \cdot A_a^{-1} = I + \varepsilon$$

ε MATRIZ SUPUESTAMENTE PEQUEÑA !!!
EN CUALQUIER CASO FACIL DE OBTENER

PROPONEMOS

$$A^{-1} = A_a^{-1} (I + \delta)$$

PREMULTIPLICANDO POR A

$$I = (I+\varepsilon)(I+\delta) \simeq I + \varepsilon + \delta$$

$$\varepsilon = -\delta \qquad \delta = -\varepsilon$$

CON LO CUAL RECUPERAMOS

A^{-1} REPETIR

Sea el sistema

$$A \cdot \bar{X} = \bar{b}$$

LA ESTABILIDAD DE SU SOLUCIÓN:
COMPORTAMIENTO DE LA SOLUCIÓN $\bar{X}$
FRENTE A POSIBLES ERRORES
EN EL VECTOR DATO $\bar{b}$

ESTOS ERRORES PUEDEN
—GENERALMENTE OCURRE—
AMPLIFICARSE EXTRAORDINARIAMENTE
INVALIDANDO LA SOLUCION
NUMERICA

$$\bar{X} = A^{-1} \cdot \bar{b}$$

DEPENDE FUERTEMENTE DE LOS
VALORES PROPIOS DE LA MATRIZ

$$\tilde{A} \cdot A$$

PODRÍA PENSARSE QUE SI

$$\det A \to 0$$

TENDREMOS PROBLEMAS EN LA RESOLUCIÒN.

PERO, ASEGURADO QUE EL PROBLEMA ES RESOLUBLE

LOS PROBLEMAS DE ESTABILIDAD PROVIENEN DE LOS VALORES PROPIOS DE $\tilde{A} \cdot A$.

SEAN $\bar{a}_k$, α_k LOS VECTORES Y LOS VALORES PROPIOS DE A

COMO A PUEDE SER CUALQUIERA (REAL)

α_k PUEDEN SER COMPLEJOS

$\bar{a}_k$ PUEDEN NO SER ORTOGONALES

Y, COMO CONSECUENCIA NO SE PUEDE ESTUDIAR NADA

RESOLVEREMOS (ESTUDIAREMOS)

$$\tilde{A} \cdot A \, \bar{x} = \tilde{A} \cdot \bar{b} \equiv \bar{B}$$

AHORA $\tilde{A} \cdot A$ ES SIMETRICA

Y TAL QUE TIENE:

VALORES PROPIOS μ_k^2

RELES POSITIVOS

VECTORES PROPIOS $\bar{u}_k$

ORTOGONALES

PUEDEN FORMAR

UNA BASE

DESCOMPONGAMOS EL VECTOR

DATO $\bar{B}$ EN ESTA BASE:

$$\bar{B} = \sum_k \beta_k \, \bar{u}_k$$

NO HAY PROBLEMAS

OPERACIÓN SIEMPRE POSIBLE

Y COMODA

ENTONCES, PROPONDREMOS PARA LA SOLUCIÓN $\bar{X}$ UN DESAROLLO DEL TIPO

$$\bar{X} = \sum_k \gamma_k \, \bar{u}_k$$

SERÁ:

$$\gamma_k = \frac{\beta_k}{\mu_k^2}$$

CON LO CUAL, SI μ_k^2 ES MUY PEQUEÑO, Y β_k TIENE ERRORES

PODEMOS TENER PROBLEMAS GRAVES

SI $$\bar{B} = \sum_k (\beta_k + \varepsilon_k) \, \bar{u}_k$$

$$\bar{X} = \sum_k \frac{(\beta_k + \varepsilon_k)}{\mu_k^2} \, \bar{u}_k$$

PUEDEN AMPLIFICARSE MUCHISIMO LOS ERRORES QUE CORRESPONDEN AL COEFICIENTE β_k PARA EL CUAL μ_k^2 ES MINIMO.

As Fadeev[44, 45] has shown, the determinacy of the system (8) is closely connected with the set of eigenvalues μ_K of the matrix K^*K [K is the matrix of the system (8), and K^* is the transposed matrix].

The determinacy (stability) decreases with increase of the ratio μ_{max}/μ_{min}. The solution "shifts about" in the directions of the eigenvectors ϕ_K of the matrix K^*K that correspond to the smaller eigenvalues μ_K. More exactly, the sensitivity of the projection of the solution vector φ in the direction of ϕ_K to variations of the components of the vector f and of the elements of the matrix K is proportional to μ_K^{-1}. Therefore, in particular, the demands on computational accuracy increase rapidly with the ratio μ_{max}/μ_{min} even for an exactly known vector f.

AMPLIFICACION DE LOS POSIBLES ERRORES

DATOS $\quad \bar{B} = \bar{B}_0 + \bar{\Sigma}$

$\bar{B}_0$: VALOR CORRECTO (MEDIO) DE $\bar{B}$

Σ ERRORES ALEATORIOS

$$\|\bar{B}_0\| = \sqrt{\sum_k \beta_k^2}$$

$$\|\bar{\Sigma}\| = \sqrt{\sum_k \Sigma_k^2}$$

$\beta_k \quad \Sigma_k$

COMPONENTES SEGUN $\bar{u}_k$ (ORTOGONALES)

$$\bar{X} = \bar{X}_0 + \bar{\delta} = \sum_k \frac{(\beta_k + \Sigma_k)}{\mu_k^2} \bar{u}_k$$

$$\bar{X}_0 \equiv \sum_k \frac{\beta_k}{\mu_k^2} \bar{u}_k \qquad \bar{\delta} \equiv \sum_k \frac{\Sigma_k}{\mu_k^2} \bar{u}_k$$

$$\|\bar{X}_0\| = \sqrt{\sum_k \frac{\beta_k}{\mu_k^4}} \qquad \|\bar{\delta}\| = \sqrt{\sum_k \frac{\Sigma_k^2}{\mu_k^4}}$$

COMO ERA $\quad \gamma_k \equiv \dfrac{\beta_k}{\mu_k^2}$

$$\|\bar{X}_0\| = \sqrt{\sum_k \gamma_k^2} \qquad \|\bar{B}_0\| = \sqrt{\sum_k \gamma_k^2 \mu_k^4}$$

AMPLIFICACION DEL ERROR

$$\frac{\dfrac{\|\bar{\delta}\|}{\|\tilde{x}_0\|}}{\dfrac{\|\bar{\varepsilon}\|}{\|B_0\|}} = \sqrt{\frac{\left(\sum_k \dfrac{\varepsilon_k^2}{\mu_k^4}\right)\left(\sum_k \gamma_k^2 \mu_k^4\right)}{\left(\sum_k \varepsilon_k^2\right)\left(\sum_k \gamma_k^2\right)}}$$

DEPENDIENDO DE LOS

COEFICIENTES γ_k

Y DE LOS VALORES PROPIOS

μ_k^2

DE LA MATRIZ $\tilde{A}.A$

EL PEOR FACTOR DE

AMPLIFICACION DEL ERROR ES

$$\sqrt{\frac{\mu_{MAX}^2}{\mu_{MIN}^2}}$$

DEPENDIENDO DE

LA RELACIÓN ENTRE LOS COEFICIENTES β_K

LA RELACION ENTRE LOS ERRORES ε_K DE CADA UNO DE ELLOS.

LA RELACIÓN ENTRE LOS AUTOVALORES μ_K^2

EL ERROR RELATIVO

SOBRE TODO EL ERROR RELATIVO QUE CORRESPONDE A LOS COEFICIENTES β_K PARA LOS CUALES μ_K^2 ES MAS PEQUEÑO

SE PUEDE AMPLIFICAR MUCHÍSIMO.

CONCLUSIONES

This analysis shows that the critical quantity which decides the physical reliability of a strictly mathematical solution is not the determinant of the system, but the *ratio of the largest to the smallest eigenvalue of the symmetrized matrix* $\tilde{A}A^1$. It is the *square root* of this ratio which measures the magnification of the noise in the direction of the smallest eigenvalue. As long as this ratio does not increase above a certain danger point, the problem of noise is not critical. But if that ratio becomes 10^4 and more, magnification of the noise in the direction of the smallest eigenvalue becomes 100 and more. The accuracy of our physical measurements is seldom sufficient to tolerate such an increase of the noise in certain directions. Any linear system whose critical ratio surpasses 10^4 can hardly be considered adequate for full determination of the unknowns of the problem.

If we do not go into the detailed analysis of the noise problem by finding the eigenvalues and eigenvectors of the matrix $\tilde{A}A$, it is still imperative that we should convince ourselves that the physical noise will not drown out our alleged solution. For this purpose we modify the given right side by random quantities of the order of magnitude of the errors of the measurements and observe the influence of this modification on our solution. If the solution changes by too large amounts as the result of this perturbation, we must come to the conclusion that our solution, although mathematically correct, cannot be considered an adequate solution of the given physical problem.

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ES ALGO DE LO MÁS IMPORTANTE QUE HE LEIDO NUNCA

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LA ESTABILIDAD DEL SISTEMA

$$\underline{\underline{K}}\ \bar{f} = \bar{q}$$

DEPENDE DE LOS AUTOVALORES $\{\mu_j\}$ DE LA MATRIZ

$$\underline{\underline{K}}^T \cdot \underline{\underline{K}}$$

LA ESTABILIDAD DECRECE AL INCREMENTAR EL COCIENTE

$$\frac{\mu_{MAX}}{\mu_{MIN}}$$

ENTONCES SE PLANTEA EL PROBLEMA DE SI ES MEJOR RESOLVER EL SISTEMA

$$\underline{\underline{K}}\ \bar{f} = \bar{q}$$

O RESOLVER EL SISTEMA

$$\underline{\underline{K}}^T\ \underline{\underline{K}}\ \bar{f} = \underline{\underline{K}}^T\ \bar{q}$$

LOS AUTOVALORES DE ESTE SEGUNDO SISTEMA SON DEL ORDEN DEL CUADRADO DE LOS DEL PRIMERO.

EL CORRESPONDIENTE μ_{MAX}
ES MAYOR QUE EN EL PRIMER CASO

EL μ_{MIN} MENOR

EL CRITERIO DE ESTABILIDAD
PEOR (*)

PERO EN EL SEGUNDO CASO LOS AUTOVALORES SON REALES Y POSITIVOS (LO QUE PUEDE FAVORECER LA INVERSIÓN) MENOS ERRORES DEBIDOS AL PROCESO DE INVERSIÓN

(*) PERO LA CRITICA ES ABSURDA
YA QUE NO SE PUEDE HABLAR DE MAYORES O MENORES AUTOVALORES EN

$$\underline{\underline{K}}\,\bar{f} = \bar{q}$$

PUEDEN SER COMPLEJOS

(No sirve el criterio)