7

PROBLEMAS MAL PLANTEADOS

PROBLEMAS MAL CONDICIONADOS

7.1

SITUACIÓN GENERAL

We shall now see a set of rules guiding the study of an Inverse Problem as a whole, a set some times called the mathematical "Inversion Theory". The remarks that are necessary to relate this strategy to all other possible information sources completes the Inversion Theory and will be seen later.

The first foundation of Inversion Theory is the notion of a well posed problem.

Formally, to solve an inverse problem means to discover the cause of a known result. Hence, all problems of the interpretation of observed data are actually inverse.

At present, a good deal of all computer activity is related to the mathematical simulation of natural objects. Modern computers can successfully solve the direct problems (that is, calculate the observable properties) for very complicated models. But when inverse problems are attempted very severe difficulties are often encountered.

La relación que existe entre un sistema físico caracterizado por un conjunto de funciones $\{f(y)\}$ y/o de variables $\{x\}$, y la información que de él tenemos vía medida de un conjunto de funciones $\{g(y)\}$, y/o de variables $\{y\}$, se suele representar por

$$O(x,y) \otimes f(x) = g(y)$$

donde el operador $O(x,y)$ representa el proceso que sintetiza, a partir de las funciones y variables del sistema, las funciones y variables medidas.

S. TWOMEY

INTRODUCTION TO THE MATHEMATICS OF INVERSION IN REMOTE SENSING AND INDIRECTED MEASUREMENTS.

ELSEVIER SCIENTIFIC PUBL. AMSTERDAM 1977

Inversion problems have existed in various branches of physics and engineering for a long time, but in the past ten or fifteen years they have received far more attention than ever before.

EN LO QUE CONCIERNE A LOS MEDIOS PARA TRABAJAR

The reason, of course, was the arrival on the scene of large computers (which enabled hitherto abstract algebraic concepts such as the solution of linear systems of equations in many unknowns to be achieved arithmetically with real numbers)

EN LO QUE CONCIERNE A LOS FINES

LA EXISTENCIA DE NUEVAS NECESIDADES: EL ESTUDIO DE OBJETOS SOBRE LOS QUE NO SE PUEDE ACTUAR.

BIEN PORQUE ESTÁN LEJOS

- the launching of earth-orbiting satellites which viewed the atmosphere from outside and which could only measure atmospheric parameters remotely and indirectly by measuring electro-magnetic radiation of some sort and relating this to the atmospheric parameter being sought, very often implying a mathematical inversion problem of some sort.

- OBJETOS ASTRONOMICOS

BIEN POR NO PODER O NO QUERER DESTRUIRLOS

- GEOFÍSICA
- MEDICINA: ESCANER: TOMOGRAFÍA COMPUTARIZADA

It was soon found that the dogmas of algebraic theory did not necessarily carry over into the realm of numerical inversion and some of the first numerical inversions gave results so bad — for example, in one early instance negative absolute temperatures — that the prospects for the remote sensing of atmospheric temperature profiles, water vapor concentration profiles, ozone profiles, and so on, for a time looked very bleak, attractive as they might have appeared when first conceived.

For a time it was thought that precision and accuracy in the computer were the core of the problem and more accurate numerical procedures were sought and applied without success. Soon it was realized that the problem was not inaccuracy in calculations, but a fundamental ambiguity in the presence of inevitable experimental inaccuracy — sets of measured quantities differing only minutely from each other could correspond to unknown functions differing very greatly from each other. Thus in the presence of measurement errors (or even in more extreme cases computer roundoff error) there were many, indeed an infinity of possible "solutions". Ambiguity could only be removed if some grounds could be provided, from outside the inversion problem, for selecting one of the possible solutions and rejecting the others. These grounds might be provided by the physical nature of the unknown function, by the likelihood that the unknown function be smooth or that it lie close to some known climatological average. It is important, however, to realize that most physical inversion problems are ambiguous — they do not possess a unique solution and the selection of a preferred unique solution from the infinity of possible solutions is an imposed additional condition. The reasonableness of the imposed condition will dictate whether or not the unique solution is also reasonable. There have been many advances in mathematical procedures for solving inversion problems but these do not remove the fundamental ambiguity (although in some instances they very effectively hide it). Our measurements reduce the number of possible solutions from infinity to a lesser but still infinite selection of possibilities. We rely on some other knowledge (or hope) to select from those a most likely or most acceptable candidate.

S. TWOMEY

3 transparencias separadas

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Thus in the presence of measurement errors (or even in more extreme cases computer roundoff error) there were many, indeed an infinity of possible "solutions". 2

1 + AMBIGUITY
PRESENCE OF INEVITABLE
EXPERIMENTAL INACCURACY

2 COMPUTER ROUNDOFF ERRORS
(IMPRECISIONES EN LAS OPERACIONES
NUMERICAS DEL ALGORITMO)

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(*) INTRINSECAMENTE AMBIGUOS pero sobre todo NUMERICAMENTE AMBIGUOS

There have been many advances in mathematical procedures for solving inversion problems but these do not remove the fundamental ambiguity (although in some instances they very effectively hide it).

MUCHAS DIFICULTADES Y AMBIGÜIDADES, SON INTRÍNSECAS DE LOS P. I.: NO DEPENDEN DEL METODO DE SOLUCIÓN.

PERO NO OLVIDAR QUE ESTE MÉTODO DE SOLUCION PUEDE, SÍ NO SUPRIMIR LAS AMBIGÜIDADES INTRÍNSECAS, SÍ AÑADIR INESTABILIDADES Y DIFICULTADES

PARKER

The theory falls into two distinct parts. One deals with the ideal case in which the data are supposed to be known exactly and as densely as desired, and is of course mainly the province of the applied mathematician. The other treats the practical problems that are created by incomplete and imprecise data. It might be thought that an exact solution to an inverse problem with perfect data would prove extremely useful for the practical case. Actually, this often turns out to be untrue because geophysical inverse problems are almost always unstable in a sense to be defined more precisely later; when this is so, the solution obtained by the analytic technique is very sensitive to the way in which the data set is completed and to the errors in it. In my view analytical studies are more valuable for their results concerning uniqueness and conditions for existence and stability.

The advances mentioned earlier arise primarily from a recognition that practical inverse problems never possess unique solutions and that an honest attempt to interpret the data must appraise the variety of compatible solutions. In the class of linear problems there is now a very satisfactory body of theory for achieving this end, but work on the larger and more prevalent class of nonlinear problems is still in a relatively primitive state and perhaps always will be.

ES VERDAD QUE ES MUY DIFICIL COMPATIBILIZAR SOLUCIONES ANALITICAS CON DATOS PRACTICOS: OBSERVACIONALES O EXPERIMENTALES.

PERO LAS SOLUCIONES ANALITICAS NOS PUEDEN AYUDAR MUCHO A COMPRENDER LAS DIFICULTADES DEL PROBLEMA Y A ENCONTRAR METODOS DE SOLUCIÓN OPTIMOS DE ACUERDO CON LA IMPRECISIÓN INTRÍNSECA Y LA DE LOS DATOS ~~NAT~~URARLES - PRACTICOS

PARKER pg. 39-40

Weidelt (1972) employs a modification of the Gel'fand-Levitan method (1955). This method has the advantage of a degree of generality, for inverse problems where unknown parameters appear as coefficients in a Sturm-Liouville differential equation can usually be cast into the required form. A Fredholm integral equation is derived from the spectrum of the differential operator. Weston (1972) uses the Gel'fand-Levitan approach, but on data recorded in the time domain, rather than using the frequency-domain Fourier transform.

Having applied some of these methods to field observations, Bailey (1973) concluded that they were not particularly successful. The reason for this, in simple terms, is that the relevant equations describe magnetic and electric fields diffusing into the Earth, and that the information about deep structures is returned to the surface by strongly attenuated fields. Thus, unless the measurements are of astronomical precision, that information is lost. The analytic solutions seem to rely heavily upon the infinite density and precision of the idealized data and are therefore unsuitable for application to actual measurements.

ES LA CONCLUSIÓN DE PARKER
EN LO QUE SE REFIERE A LA
APLICACIÓN DE LOS METODOS DE
SOLUCION TEORICA DEL PROBLEMA:
CALCULO DE POTENCIALES (COEFICIENTES)
DE LAS ECUACIONES QUE DESCRIBEN
LA PROPAGACIÓN DE ONDAS ELECTRO-
-MAGNETICAS EN MEDIOS DISPERSIVOS
APLICADAS AL ESTUDIO DE
LA ESTRUCTURA DE LA TIERRA

SI LA MAYORÍA DE ESTOS PROBLEMAS CON INESTABILIDADES MANIFIESTAS HAN APARECIDO SOLAMENTE EN LA ÉPOCA DE LOS ORDENADORES ($T > 1960$), ES POR DOS RAZONES FUNDAMENTALES:

- LOS GRANDES SISTEMAS LINEALES SÓLO HAN PODIDO RESOLVERSE CON EL ORDENADOR. COMO CONSECUENCIA APARECEN PROBLEMAS QUE EN LOS SISTEMAS PEQUEÑOS ERA OBVIO ESTUDIAR A PRIORI, O EN EL MOMENTO DE RESOLVERLOS

  PERO LOS PROBLEMAS ERAN CONOCIDOS, SÓLO QUE UNO NO ESPERA QUE LE OCURRAN A ÉL

- LAS ECUACIONES TRANSCENDENTES SE RESOLVÍAN GRÁFICAMENTE Y CALCULANDO "A MANO". ENTONCES ERA OBVIO EL DARSE CUENTA DEL COMPORTAMIENTO DE UNA ECUACIÓN ALREDEDOR DE LA SOLUCIÓN (ESTABILIDAD).

  SE PODÍA VER SI $f(x) = 0$ SE CUMPLÍA NETAMENTE PARA $x = x_0$, O BIEN ASINTÓTICAMENTE. EN ESTE ÚLTIMO CASO SE ERA CONSCIENTE DE LA IMPOSIBILIDAD PRÁCTICA DE SOLUCIÓN

LO MISMO OCURRÍA CON VARIAS ECUACIONES: DIBUJANDO LAS CURVAS CORRESPONDIENTES, ERA FACIL EL DARSE CUENTA DE LAS CONDICIONES DE ORTOGONALIDAD / NO ORTOGONALIDAD ENTRE ELLAS

AHORA, CON LOS ORDENADORES, AL AUMENTAR LA DIMENSIÓN DE LOS PROBLEMAS, AUMENTAN LAS DIFICULTADES, O MEJOR, LA PROBABILIDAD DE QUE OCURRAN DIFICULTADES

PERO, ADEMAS, NO SE VEN DIRECTAMENTE, A NO SER QUE EL RESULTADO SEA CATASTROFICO

AHORA, SE PROGRAMA UN CÁLCULO, SE OBTIENE UN VALOR PARA LOS RESULTADOS
SE ANOTA, Y VALE.
NO SE SABE NADA MAS

SALVO QUE A OTRO SE LE OCURRA REPETIR EL CALCULO Y VARÍE LIGERAMENTE LAS CONDICIONES, Y, COMO CONSECUENCIA OBTENGA UN RESULTADO COMPLETAMENTE DIFERENTE

ANTES SE ERA CONSCIENTE "A PRIORI" DE TODA ESTA PROBLEMATICA.

ENTRE OTRAS RAZONES PORQUE ERA "MUY DURO" TRABAJAR "A MANO"

AL RECHAZAR, POR DIFICULTADES PRACTICAS, MUCHOS PROBLEMAS DE GRAN DIMENSIÓN

SE OBVIARON DIFICULTADES QUE APARECEN ACTUALMENTE.

S. TWOMEY

$$g(y) = \int_a^b k(x, y)\, f(x)\, dx$$

The crux of the difficulty was that numerical inversions were producing results which were physically unacceptable but were mathematically acceptable (in the sense that *had* they existed they would have given measured values identical or almost identical with what was measured). There were in fact ambiguities — the computer was being "told" to find an $f(x)$ from a set of values for $g(y)$ at prescribed values of y, it was "told" what the mathematical process was which related $f(x)$ and $g(y)$, but it was not "told" that there were many sorts of $f(x)$ — highly oscillatory, negative in some places, or whatever — which, either because of the physical nature of $f(x)$ or because of the way in which direct measurements showed that $f(x)$ usually behaved, would be rejected as impossible or ridiculous by the recipient of the computer's "answer". And yet the computer was often blamed, even though it had done all that had been asked of it and produced an $f(x)$ which via the specified mathematical process led to values of $g(y)$ which were exceedingly close to the initial data for $g(y)$ supplied to the computer. Were it possible for computers to have ulcers or neuroses there is little doubt that most of those on which early numerical inversion attempts were made would have acquired both afflictions.

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7. 2

ESQUEMA DE LA PROBLEMATICA

Supongamos que EL MODELO DE UN FENOMENO, SE REPRESENTA POR UN OPERADOR K, ACTUANDO SOBRE UN VECTOR $\vec{F}$:

$$\bar{F} = \{F(x_i)\}$$

UN CONJUNTO DE VALORES $\{F(x_i)\}$ que PERTENECEN A EL ESPACIO DE LOS OBJETOS E_F

LA ACTUACIÓN DE K SOBRE $\vec{F}$ CONDUCE A UN VECTOR $\bar{G}$

$$K \otimes \bar{F} = \bar{G}$$

DONDE $\bar{G} \equiv \{G(y_j)\}$ ES UN CONJUNTO DE VALORES QUE PERTENECEN AL ESPACIO DE LAS IMAGENES E_G

K REPRESENTA UN SISTEMA DE ECUACIONES ALGEBRAICAS O INTEGRO-DIFERENCIALES.

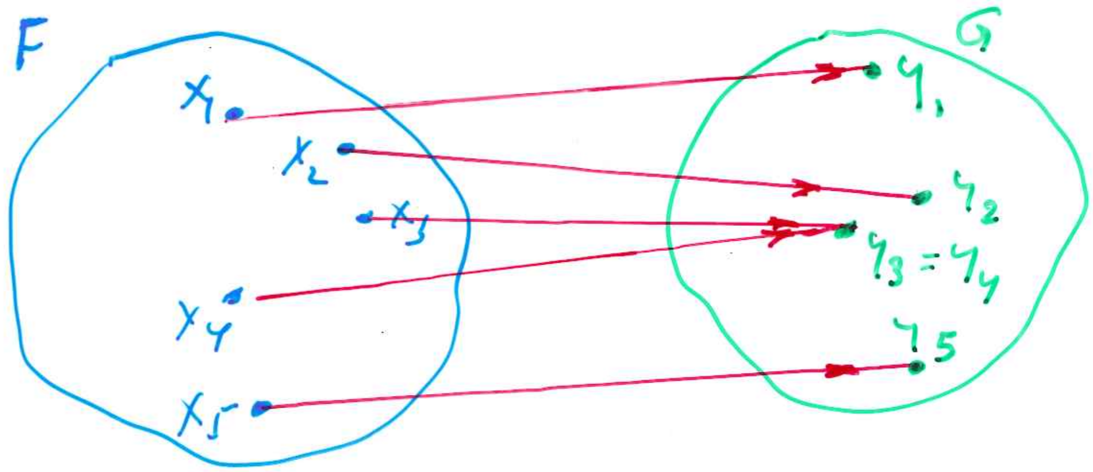
SE CONOCEN LOS VALORES DE $\bar{G} = \{G(y_j)\}$

SE QUIEREN DETERMINAR LOS VALORES DE $\bar{F} \equiv \{F(x_i)\}$

$O(x, y) \propto \left\{ \begin{matrix} x \\ f(x) \end{matrix} \right\} = \left\{ \begin{matrix} y \\ g(y) \end{matrix} \right\}$

$O(x, y)$ PROYECCIÓN DEL ESPACIO F (PARAMETROS) SOBRE EL ESPACIO G (VARIABLES)

PUEDE OCURRIR QUE DOS (Ó MAS) PUNTOS DEL ESPACIO F (parámetros) PROYECTEN SOBRE EL MISMO PUNTO DEL ESPACIO G (variable)



MUCHOS PROBLEMAS INVERSOS SON MAL PLANTEADOS EN EL SENTIDO DE QUE PARA UN MISMO CONJUNTO DE DATOS: VARIABLES $\{y_k\}$ EXISTIRÁN VARIAS (a veces infinitas) SOLUCIONES

Ejemplo $\quad X^2 = y$

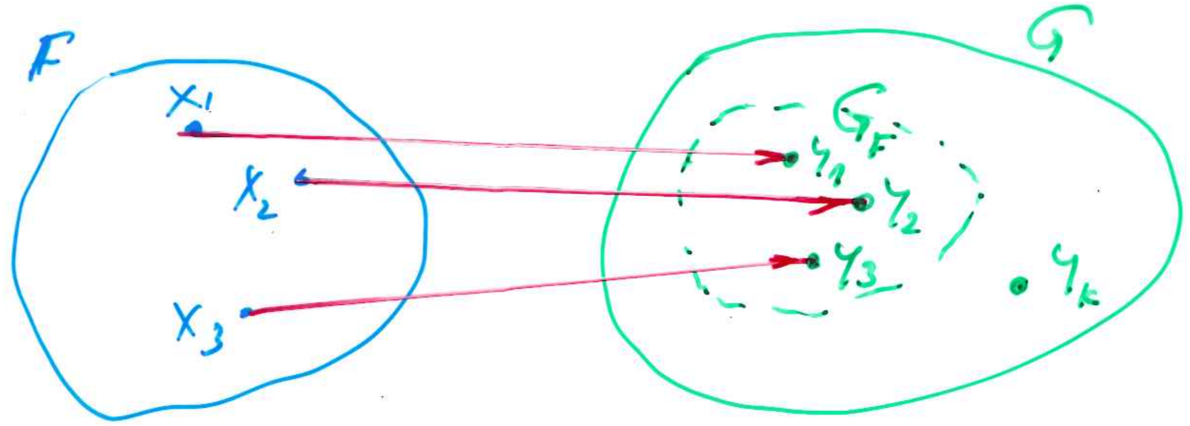
$X = 1 \xrightarrow{P.D.} y = 1$

$X = -1 \xrightarrow{P.D} y = 1$

$y = 1 \xrightarrow{P.I} ?$

OTRA POSIBILIDAD NEGATIVA (DESGRACIADAMENTE MUY FRECUENTE)

PUEDE OCURRIR QUE EL CONJUNTO G_F SOBRE EL QUE PROYECTA EL ESPACIO GLOBAL F DE LOS POSIBLES VALORES DE LOS PARAMETROS, SEA MENOR QUE EL ESPACIO GLOBAL G DE LAS VARIABLES MEDIDAS: DATOS DEL P.I.



EN ESTAS CONDICIONES, PARA LOS DATOS y_k QUE PERTENECEN A G PERO NO A G_F, NO EXISTE SOLUCIÓN

Ejemplo $\qquad x^2+1=y$

PARA DATOS $\qquad y<1$

NO EXISTE SOLUCIÓN REAL

PERO... ¿DE DONDE HEMOS SACADO ESOS DATOS?

PROBLEMAS LINEALES

NO SE PLANTEARÁ PUES LA PRIMERA POSIBILIDAD ANTERIOR

EN PURA TEORÍA PARA CADA FAMILIA DE DATOS: VARIABLES OBSERVADAS HABRÁ UNA, Y SÓLO UNA SOLUCIÓN.

PERO DEBIDO A IMPRECISIONES EN LA MEDIDA DE LOS VALORES DE ESAS VARIABLES OBSERVADAS

IN PRINCIPLE, ZERO-ERROR MEASUREMENTS DO NO EXIST

NOS ENCONTRAREMOS MUY FRECUENTEMENTE EN EL CUADRO DE LA SEGUNDA POSIBILIDAD:

LOS DATOS PUEDEN ESTAR FUERA DEL ESPACIO G_F DONDE PROYECTAN TODOS LOS POSIBLES VALORES DE LOS PARAMETROS

ENTONCES NO EXISTIRA SOLUCIÓN.

Pero ...

Pero..

COMO EL ORDENADOR NO ENTIENDE DE ESPACIOS DONDE PROYECTAN LOS OPERADORES (y muchos astrónomos tampoco) PUEDE QUE NOS PROPORCIONE UN RESULTADO ABERRANTE

Y LO PEOR ES QUE LO INTERPRETEMOS COMO UNA POSIBILIDAD FISICA

VEREMOS SITUACIONES DE ESTE TIPO.

7-3

PROBLEMAS BIEN/MAL PLANTEADOS

HADAMARD

J. HADAMARD en 1932

LE PROBLÈME DE CAUCHY ET LES ÉQUATIONS AUX DÉRIVÉES PARTIELLES LINEAIRES HYPERBOLIQUES

Paris, Hermann

DEFINE UN PROBLEMA BIEN PLANTEADO COMO AQUÉL EN DONDE

1. LA ECUACION $Kx = y$ TIENE SOLUCIÓN PARA CUALQUIER VALOR DE

2. ESTA SOLUCIÓN ES UNICA

3. LA SOLUCIÓN ES ESTABLE; es decir: PERTURBACIONES PEQUEÑAS DE y DEBEN IMPLICAR PERTUBACIONES PEQUEÑAS EN LA SOLUCIÓN x

From

I.J.D. CRAIG and J.C. BROWN 1986.

"INVERSE PROBLEMS IN ASTRONOMY" (pag. 11)

Adam Hilger Ltd. Bristol and Boston

These two formulations of this simple case establish at least in an illustrative way that the inversion of integral equations can be 'ill posed' in the sense that small changes in the data can give rise to large changes in the solution.

Readers without practical experience of such problems may, however, still be left with the impression that there is no fundamental problem here because they believe that uncertainties in their data can ultimately always be made as small as they want.

The view is fallacious. Numerical represen-tations of data always contain inescapable sources of uncertainty for, in addition to random errors and systematic instrumental effects, the nature of mea-surement itself introduces uncertainties or ambi-guities by discretisation of a continuum and, in many cases, by truncation of the domain of the data function to some restricted domain over which data can be taken.

IN PRINCIPLE
ZERO-ERROR MEASUREMENTS
DO NO EXIST

PERO SI TRATAMOS UN
PROBLEMA QUE NO SEA MUY ESTABLE

Mathematically, a problem is said to be stable if the solution depends continuously on the data and unstable if it does not. In simple terms this means that for all data sets lying close[2] to a particular set the solutions fall close to each other.

LA SITUACIÓN PUEDE SER MUY
COMPLICADA: LOS RESULTADOS
TOMARÁN (PUEDEN TOMAR) CUALQUIER
VALOR

Moreover, the severity of the instability is so great for many important integral deconvolutions that the gain in information on the desired source function is very small even for very large improvements in data accuracy, if such are achievable at all.

It was however soon discovered that many of the problems of science and technology went beyond the domain of the theoretical mathematics as based on the concept of "absolute precision" that lies at the heart of Aristotelian logic. Typical of this sort of problem are the processing of experimental data and optimal planning.

For a problem to be tractable by a computer treatment, the following components must be present: (i) a mathematical model of the problem, (ii) a stable method for solving the corresponding mathematical problem, (iii) a stable algorithm for this method, and (iv) computer code implementing the algorithm.

The solution of an unstable problem when the initial data is approximated can change substantially even if the variation in the data is arbitrarily small. Thus, if implemented on a computer, the algorithm to solve such a problem yields unstable results.

That is, computer solutions of unstable problems within the framework of the "precision concept" of theoretical mathematics do not ensure stable output results.

Solving unstable problems with approximate initial data by computer involves methods that belong to a class of mathematical problems which lies outside the domain of theoretical mathematics. Such problems often appear in scientific and technological applications.

ASÍ, DENTRO DE ESTA EXPOSICIÓN DEL CONCEPTO DE P.I. APARECEN

ADEMÁS DE LA NECESIDAD DE UN MODELO MATEMÁTICO QUE RELACIONE LAS VARIABLES MEDIDAS CON LOS PARÁMETROS QUE QUEREMOS DETERMINAR

- LA NECESIDAD DE UN MÉTODO PARA RESOLVER EL CORRESPONDIENTE PROBLEMA MATEMÁTICO

- UN ALGORITMO PARA DESARROLLAR ESE MÉTODO

- CODIGO NUMERICO PARA IMPLEMENTAR EL ALGORITMO

SÍ QUEREMOS QUE UN PROBLEMA SEA ESTABLE

ADEMÁS DE LA ESTABILIDAD INTRINSECA DEL PROPIO PROBLEMA

DEBEN DE SER ESTABLES

- EL MÉTODO PARA SU RESOLUCIÓN
- EL ALGORITMO PARA DESARROLLAR EL MÉTODO
- EL CORRESPONDIENTE CODIGO DE COMPUTACIÓN

PUEDE QUE UN PROBLEMA SEA INTRINSECAMENTE ESTABLE

PERO ESTAR PRACTICAMENTE MAL PLANTEADO

GENERALMENTE SE OLVIDAN ESTOS CONSEJOS.

LA CONCLUSIÓN MAS IMPORTANTE DE TODO LO ANERIOR (y la que menos se tiene en cuenta):
PARA LOS ALGORITMOS

COMO EN EL CASO DE LOS ERRORES EN LAS MEDIDAS,

Readers without practical experience of such problems may, however, still be left with the impression that there is no fundamental problem neither here. The blame for this commonly held, but in practical terms entirely fallacious, view can probably be lodged against the classical style of most undergraduate applied mathematics courses which give the impression that the transition from functions defined on the continuum to their accurate numerical representation is a minor matter, the details of which can be left to the computing department or service.

LA IMPRECISIÓN NUMERICA
PROPIA DE UN ALGORITMO
COMPLICA LA RESOLUCIÓN
DE UN PROBLEMA INVERSO,

MUCHO MAS QUE LA
IMPRECISIÓN / ERRORES
DE LOS DATOS

MUCHAS VECES
UNA SOLUCIÓN APROXIMADA
QUE CORRESPONDE A UN ALGORITMO
APROXIMADO

QUE HA SUFRIDO CIERTAS SIMPLIFICA-
CIONES EN LOS DATOS Y EN LAS
PROPIEDADES DEL ALGORITMO QUE
CONCIERNEN LA ESTABILIDAD

AUNQUE IMPLIQUEN ALGUNAS-
-LIGERAS COMPLICACIONES DEL
TRATAMIENTO NUMÉRICO

PUEDE SER EXTRAORDINARIAMENTE
MÁS CORRECTA QUE LA SOLUCIÓN
CORRESPONDIENTE AL ALGORITMO
MATEMATICAMENTE NO SIMPLIFICADO

- TECNICAS DE ESTABILIZACIÓN:
  REGULARIZACIÓN SI SE REFIEREN
  A LA MODIFICACIÓN DEL ALGORITMO

- FILTRADO DE DATOS, SI SE REFIEREN
  A LA PREPARACION DE LOS DATOS
  DE ACUERDO CON EL PROBLEMA

VEREMOS SU NECESIDAD CUANDO
ESTEMOS MAS AVANZADOS

UN EJEMPLO MUY IMPORTANTE DE P.I. MUY INESTABLE: LA CONSTRUCCIÓN DE UN POLINOMIO DE ORDEN N-1 → N COEFICIENTES - QUE PASA POR N PUNTOS (x_j, y_j) DADOS

POLINOMIO DE INTERPOLACIÓN DE LAGRANGE

DATOS N PUNTOS (x_j, y_j)

PARAMETROS QUE HAY QUE DETERMINAR: N COEFICIENTES

$$P(x) = \sum_{j=1}^{N} y_j \prod_{\substack{k=1 \\ k \neq j}}^{N} \frac{x - x_k}{x_j - x_k}$$

EVIDENTEMENTE $P(x_j) = y_j$

LA EXPRESIÓN ANTERIOR PROPORCIONA FACILMENTE LOS COEFICIENTES

PERO SÍ VARIAMOS LIGERAMENTE LOS DATOS: AHORA $(x_j + \varepsilon, y_j + \delta)$

LOS COEFICIENTES (Y LOS VALORES DEL POLINOMIO EN OTROS PUNTOS) PUEDEN SER COMPLETAMENTE DIFERENTES

SIN EMBARGO, ADMITIENDO QUE LOS N PUNTOS (X_j, Y_j) PUEDEN TENER INDETERMINACIONES Y ERRORES EN SUS VALORES

PODEMOS APROXIMARNOS A ELLOS MEDIANTE UN POLINOMIO DE ORDEN MENOR QUE N-1 CON UN CRITERIO DE APROXIMACION DE LOS VALORES $P(X_j)$ A LOS VALORES DE Y_j, DE ACUERDO A MINIMOS CUADRADOS (METRICA DISCRETA O CONTINUA)

TENDREMOS UN POLINOMIO DE INTERPOLACIÓN ABSOLUTAMENTE ESTABLE, QUE SE AJUSTA CON EL CRITERIO ANTERIOR A LOS PUNTOS (X_j, Y_j) MEDIDOS

EN CIERTO MODO:

HEMOS "FILTRADO LOS DATOS"

P.I.

MAL PLANTEADO

TIENE QUE VER CON LAS CUESTIONES DE EXISTENCIA
UNICIDAD

DE LAS SOLUCIONES

MAL CONDICIONADO

TIENE QUE VER CON LA ESTABILIDAD

VAMOS A TRABAJAR EN ESTE CURSO CON PROBLEMAS LINEALES

LUEGO, EN PRINCIPIO - Y SALVO

PROBLEMAS CON LOS DATOS: ERRORES O/Y IMPRECISIONES

LAS CUESTIONES DE EXISTENCIA Y UNICIDAD NO PLANTEARÁN MUCHOS PROBLEMAS

PERO LA CUESTIÓN DE LA ESTABILIDAD VA A SER MUY IMPORTANTE EN EL ESTUDIO DE LOS P.I.

AUNQUE SEAN LINEALES.

7 - 4

MÉTODOS DE SOLUCIÓN

MAL PLANTEADOS

PUEDE HABER

PROBLEMAS BIEN O MAL PLANTEADOS
BIEN O MAL CONDICIONADOS

CRITERIO DE HADAMARD

PERO SI SE RESUELVEN EMPLEANDO UN
- METODO MATEMÁTICO
- ALGORITMO MATEMÁTICO-NUMÉRICO
- CODIGO NUMERICO

MAL PLANTEADO
O MAL CONDICIONADO

EL PROBLEMA TOTAL ADQUIERE ESTAS PROPIEDADES.

ASÍ: PUEDE HABER
PROBLEMAS BIEN PLANTEADOS Y
BIEN CONDICIONADOS

Y METODOS MATEMATICOS-NUMERICOS
MAL PLANTEADOS

EL PROBLEMA NO ES MAL PLANTEADO,
ESTÁ MAL PLANTEADO

IMPORTANCIA DEL METODO DE SOLUCIÓN

EJEMPLO DE UN PROBLEMA INTRINSECAMENTE INESTABLE Y DE UN MÉTODO DE SOLUCIÓN QUE PUEDE DESESTABILIZAR, AUN MÁS, EL PROBLEMA

POLINOMIO DE INTERPOLACIÓN DE LAGRANGE:

OBTENER LOS N COEFICIENTES

$$a_0, a_1, a_2, \ldots a_{N-1}$$

DEL POLINOMIO

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{N-1} x^{N-1}$$

QUE SATISFACE LOS N PUNTOS

$$(x_J, y_J) \qquad J = 1, 2, \ldots N$$

PODRÍAMOS ESCRIBIR EL SISTEMA

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{N-1} \\ & & & & \\ 1 & x_N & x_N^2 & \cdots & x_N^{N-1} \end{bmatrix} \times \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \\ a_{N-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \\ y_N \end{bmatrix}$$

CUYA SOLUCIÓN PROPORCIONARÁ LOS COEFICIENTES $a_0, a_1, a_2 \ldots a_{N-1}$ BUSCADOS

PERO LA RESOLUCIÓN DEL SISTEMA ANTERIOR: VANDERMONDE

ES MUY INESTABLE

A MENOS QUE LA DIMENSIÓN SEA PEQUEÑA

SERÁ PUES UN MÉTODO MUY POCO ÓPTIMO PARA RESOLVER EL PROBLEMA

PERO SI APLICAMOS LA EXPRESIÓN QUE VIMOS YA PARA ESTE POLINOMIO DE INTERPOLACIÓN:

$$P(x) = \sum_{j=1}^{N} y_j \prod_{\substack{k=1 \\ k \neq j}}^{N} \frac{x - x_k}{x_j - x_k}$$

EL CÁLCULO DE LOS COEFICIENTES DE CADA POLINOMIO DE ORDEN $N-1$ (UNO PARA CADA x_j)

$$\prod_{\substack{k=1 \\ k \neq j}}^{N} \frac{x - x_k}{x_j - x_k}$$

COMO PRODUCTO DIRECTO ES TRIVIAL (algoritmicamente)

SE RESUELVE PUES EL PROBLEMA DE UNA FORMA MÁS CÓMODA Y ESTABLE

PRECÍSAMENTE

LA ESTRUCTURA FORMAL ANTERIOR

DEL POLINOMIO DE INTERPOLACIÓN
DE LAGRANGE

ES LA QUE SE UTILIZA, COMO
ALGORITMO PARTICULAR

PARA INVERTIR LA MATRIZ DE
VANDERMONDE

QUE APARECERÁ EN MUCHÍSIMOS
P.I. LINEALES.