THE RIEMANN-LEBESGUE LEMA

1-8 CLASSES OF FUNCTIONS

The inversion theorems for integral transforms of functions of a very general type are lengthy and somewhat intricate. Since we are interested primarily in applications we shall consider only functions of the type which normally arise in the analysis of physical problems.

If a function $f(x)$, of a single independent variable x, is continuous on an open interval (a,b), we say that $f(x)$ belongs to the class of functions $\mathscr{C}(a,b)$, and we write $f \in \mathscr{C}(a,b)$; if the interval on which the function is continuous is closed we write $f \in \mathscr{C}[a,b]$. If the function f is continuous over the whole real line we write $f \in \mathscr{C}(\mathbf{R})$ or $f \in \mathscr{C}(-\infty,\infty)$.

A function $f(x)$ is said to be *piecewise continuous* in an interval (a,b) if the interval can be partitioned into a *finite* number of nonintersecting intervals

$$(a,a_1), (a_1, a_2), \ldots, (a_{n-1},b),$$

in each of which the function is continuous and has finite limits as x approaches the endpoints of each of the subintervals. A function of this kind is shown in Fig. 1-1. When a function is piecewise continuous in (a,b) we write $f \in \mathscr{P}(a,b)$.

FOURIER'S INTEGRAL THEOREM

We saw in the last section by a purely formal argument that in some sense the identity

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}\, d\xi \int_{-\infty}^{\infty} f(u)e^{iu\xi}\, du \qquad (1)$$

might be valid. This identity is known as *Fourier's integral theorem*. In this section we shall show that Fourier's integral theorem holds for functions $f(x)$ which are piecewise-continuously differentiable and absolutely integrable on the whole real line.

Fourier's integral has been proved under much more general conditions on the behavior of the function $f(x)$. We shall look at some of these results in sec. 2-11, but we shall here confine our attention to establishing the theorem for the class of functions stated since that class is sufficiently wide to embrace most of the functions which arise in problems in mathematical physics.

This can be written in another way. If we put

$$F(\xi) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(u)e^{iu\xi}\, du,$$

then equation (1) states that

$$f(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} F(\xi)e^{-ix\xi}\, d\xi.$$

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2-2 FOURIER'S INTEGRAL THEOREM

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The central result is:

The Riemann–Lebesgue lemma *If* $f(t) \in \mathscr{C}[a,b]$, *where* $0 < a < b < \infty$,

$$\int_a^b f(t)\sin(\lambda t)\,dt \to 0, \qquad \int_a^b f(t)\cos(\lambda t)\,dt \to 0$$

as $\lambda \to \infty$.

We shall begin by proving the first result.

By a suitable change of variable we find that

$$\int_a^b f(t)\sin(\lambda t)\,dt = -\int_{a-\pi/\lambda}^{b-\pi/\lambda} f(\tau + \pi/\lambda)\sin(\lambda\tau)\,d\tau$$

and hence that

$$2\int_a^b f(t)\sin(\lambda t)\,dt = \int_a^b f(t)\sin(\lambda t)\,dt - \int_{a-\pi/\lambda}^{b-\pi/\lambda} f(t + \pi/\lambda)\sin(\lambda t)\,dt.$$

We may now write this result in the form

$$\int_a^b f(t)\sin(\lambda t)\,dt = \tfrac{1}{2}(I_1 + I_2 - I_3)$$

where

$$I_1 = \int_{b-\pi/\lambda}^{b} f(t)\sin(\lambda t)\,dt, \quad I_2 = \int_a^{b-\pi/\lambda} \{f(t) - f(t + \pi/\lambda)\}\sin(\lambda t)\,\mathrm{dt}$$

$$I_3 = \int_{a-\pi/\lambda}^{a} f(t + \pi/\lambda)\sin(\lambda t)\,dt.$$

Since $f(t)$ is continuous in $[a,b]$ it is bounded in that interval; suppose that $|f(t)| < M$ in $[a,b]$. Then given any arbitrary positive constant ε, we can choose a positive constant η_1 such that

$$\frac{\pi}{\lambda} < \frac{\varepsilon}{2M}, \qquad (\lambda > \eta_1)$$

so that, if $\lambda > \eta_1$,

$$|I_1| < M\pi/\lambda < \tfrac{1}{2}\varepsilon, \qquad |I_3| < M\pi/\lambda < \tfrac{1}{2}\varepsilon.$$

If $f(t)$ is continuous in a finite closed interval $[a,b]$ then it is uniformly continuous in $[a,b]$ so that, given ε we can find a positive number η_2 such that

$$|f(t + \pi/\lambda) - f(t)| < \varepsilon/(b - a), \qquad (\lambda > \eta_2).$$

Hence if $\lambda > \eta_2$,

$$|I_2| < \int_a^{b-\pi/\lambda} |f(t + \pi/\lambda) - f(t)|\, dt < \frac{b - a - \pi/\lambda}{b - a}\varepsilon$$

showing that $|I_2| < \varepsilon$.

Collecting these results together we see that if we write $\eta = \max(\eta_1,\eta_2)$, then

$$\left|\int_a^b f(t)\sin(\lambda t)\, dt\right| < 2\varepsilon, \qquad \lambda > \eta,$$

and the first result follows.

In proving this result the only properties of $\sin(\lambda t)$ which we have used are that it is continuous and that for real values of t, $\sin(\lambda t + \pi) = -\sin(\lambda t)$ and $|\sin(\lambda t)| \leqslant 1$. The function $\cos(\lambda t)$ has the same properties so that the second result can be established in exactly the same way.

Corollary 1 *If $f(t) \in \mathscr{P}[a,b]$, where $0 < a < b < \infty$,*

$$\int_a^b f(t)\begin{matrix}\sin\\ \cos\end{matrix}(\lambda t)\, dt \to 0,$$

as $\lambda \to \infty$.

We apply the Riemann-Lebesgue lemma to each of the integrals over the intervals $[a,a_1),(a_1,a_2), \ldots ,(a_{n-1},a_n),(a_n,b]$ in which $f(t)$ is continuous, and the result follows immediately.

Corollary 2 *If $f(t) \in \mathscr{P}[a,\infty)$, $a > 0$, and if $f(t) \in \mathscr{A}_1[a,\infty)$, then*

$$\int_a^\infty f(t)\begin{matrix}\sin\\ \cos\end{matrix}(\lambda t)\, dt \to 0$$

as $\lambda \to \infty$.

To prove this we may assume, without loss of generality, that $f(t)$ is continuous in $[a, \infty)$. Since $f(t)$ is absolutely integrable in the same interval, we can find a positive number N such that

$$\int_N^\infty |f(t)|\, dt < \tfrac{1}{2}\varepsilon$$

for any prescribed positive number ε.

Since

$$\begin{aligned} \left|\int_a^\infty f(t) \frac{\sin}{\cos}(\lambda t)\, dt\right| &= \left|\int_a^N f(t) \frac{\sin}{\cos}(\lambda t)\, dt + \int_N^\infty f(t) \frac{\sin}{\cos}(\lambda t)\, dt\right| \\ &\leqslant \left|\int_a^N f(t) \frac{\sin}{\cos}(\lambda t)\, dt\right| + \int_N^\infty |f(t)|\, dt \end{aligned}$$

and given ε and N we can, by the Riemann–Lebesgue lemma, find a positive number η such that

$$\left|\int_a^N f(t) \frac{\sin}{\cos}(\lambda t)\, dt\right| < \tfrac{1}{2}\varepsilon, \qquad \lambda > \eta$$

so that the result follows immediately.