NUMERICAL SOLUTION OF INTEGRAL EQUATIONS

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INTEGRAL EQUATIONS

General remarks

A functional equation in which the unknown function appears under an integral sign is called an integral equation; see, however, Zabreyko *et al.* (1975, p.1). For example, the quations

$$\frac{1}{f_0(x)} + \tfrac{1}{2} \int_0^1 \frac{x f_0(y)}{x + y}\, dy = 1 \qquad (0 \le x \le 1), \tag{1.1}$$

$$f_1(x) - \int_0^x (x - y)\, \{f_1(y)\}^2\, dy = e^{-x} \qquad (x \ge 0), \tag{1.2}$$

$$\int_{-\infty}^{\infty} \exp\{-(x-y)^2\}\, f_2(y)\, dy = e^x \qquad (-\infty < x < \infty), \tag{1.3}$$

and

$$\lambda \int_{-1}^{1} \int_{-1}^{1} |xu + yv|\, f_3(u,v)\, du\, dv = f_3(x,y) \quad (-1 < x,y < 1) \tag{1.4}$$

are all integral equations. The functions to be found are $f_0(x)$, $f_1(x)$, $f_2(x)$, and $f_3(x,y)$ respectively. If the derivative of the solution appears in the integral equation, as in

$$\{f''(x)\}^2 - \int_{-1}^{1} \sin xy\, f(y)\, dy = \cos x \quad (-1 < x < 1), \tag{1.5}$$

the equation is usually known as an integro-differential equation, and additional boundary conditions are required to determine the solution $f(x)$.

Integral equations occur in the mathematical theory for a number of branches of science. In particular, they occur in the study of acoustics, optics and laser theory, potential theory, radiative transfer theory, cardiology, and in fluid mechanics and statistics.

The solutions occur non-linearly in eqns (1.1), (1.2), and (1.5), and these equations are therefore said to be non-linear equations. The other examples are linear equations (the unknown function appears linearly). In the early parts of this book we treat linear equations (in which the solution is a function of a single variable, as in eqn (1.3)). Later in the book we shall consider more general types of integral equation.

One of our first tasks is to classify the different types of linear integral equation.

Preliminary classification of linear integral equations

A general form of linear integral equation is given by the equation

$$\alpha(x)\, f(x) - \lambda \int_a^b K(x,y)\, f(y)\mathrm{d}y = g(x) \ , \tag{1.6}$$

where the functions involved may be supposed to be complex-valued functions of real variables. We shall suppose that eqn (1.6) is valid for $a \leq x \leq b$, and we shall suppose, in general, that a and b are finite. The functions $\alpha(x)$, $g(x)$, and $K(x,y)$ are known for $a \leq x, y \leq b$, and λ is a constant (which, when its value is known, is sometimes absorbed in $K(x,y)$). The function $K(x,y)$ is called the *kernel* of the integral equation.

We shall consider some particular cases of (1.6). The first of these is

$$\int_a^b K(x,y)\, f(y)\,dy = g(x) \qquad (c \le x \le d)\ , \tag{1.7}$$

which is called an *equation of the first kind*. We can ensure, by a change of variable, that $c=a$ and $d=b$, given that $|abcd|$ is finite.

Example 1.1

The equation

$$\int_0^1 (x^2 + y^2)^{\frac{1}{2}} f(y)\,dy = \tfrac{1}{3}\{(1+x^2)^{\frac{3}{2}} - x^3\} \quad (0 \le x \le 1)$$

has a solution $f(x) = x$. *

The equation

$$f(x) - \lambda\int_a^b K(x,y)\, f(y)\,dy = g(x) \qquad (a \le x \le b)\ , \tag{1.8}$$

where λ is a *known* constant, is called an *equation of the second kind*.

Example 1.2

The equation

$$f(x) - \int_0^1 e^{xy} f(y)\,dy = 1 - \frac{1}{x}(e^x - 1) \quad (0 \le x \le 1)$$

has the unique solution $f(x) = 1$. *

Example 1.3

The equation

$$f(x) - 3\int_0^1 xy\, f(y)\,dy = x^2 \qquad (0 \le x \le 1)$$

has no solution. *

THE NUMERICAL TREATMENT OF INTEGRAL EQUATIONS

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Mathematical Aspects of the Inversion Problem

4.1 Introduction

It is natural to assume that the astronomical data inversion problems discussed previously can be illuminated to some extent by invoking the analytic mathematical theory of integral equations. This is indeed the case. Without attempting an exhaustive mathematical treatment, the present chapter aims to outline those aspects of the mathematical theory of relevance to practical data inversion and stabilisation techniques. The discussion is mainly concerned with integral equations of the first kind since these are known to present the most severe practical difficulties. The reader should be warned however, that no mathematical theory, no matter how powerful or elegant, can overcome the basic problem of the data generally lacking information with regard to the details—specifically the high frequency components—of the source function. Methods of stabilising the inversion by counteracting this 'lack of information' are discussed in Chapter 6.

Fuller development of material in this chapter (as well as Chapters 5 and 6) can be found in the following sources. Systematic developments of the theory of linear integral equations are presented in the textbooks by Smithies (1962) and Tricomi (1957). Of more relevance to the numerical inversion problem are the proceedings edited by Delves and Walsh (1974): the excellent contribution of Miller (Chapter 13) provides a particularly succinct account of the ill posed inversion problem. The more recent work of Baker (1977) has provided the most general and exhaustive treatment so far on the numerical treatment of integral equations. There are, in addition, many review articles that deal with different aspects and applications of the