Non-Markovian particle dynamics in continuously controlled quantum gases

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Motivation:

Localising object in lab = feedback
[Caves & Milburn, PRA 36, 5543 (1987)]

Motion of single atom in the object?

Quantum Brownian motion
Macroscopic view – Continuous feedback

Feedback loop:
- Measure centre of mass with finite resolution,
- Shift back centre of mass by a fraction.

Continuous limit: Rate of feedback $\gamma \to \infty$

Master equation for many-atom density operator (Brownian motion) [Barchielli, Nuovo Cimento B 74, 113 (1983)]:

\[
\dot{\hat{\varrho}} = -\frac{i}{\hbar} [\hat{H}, \hat{\varrho}] + i \frac{\zeta}{2\hbar} [\hat{P}, \{\hat{X}, \hat{\varrho}\}] \\
- \frac{1}{8\sigma^2} [\hat{X}, [\hat{X}, \hat{\varrho}]] - \frac{\zeta^2\sigma^2}{2\hbar^2} [\hat{P}, [\hat{P}, \hat{\varrho}]].
\]

$\sigma^2 = \sigma_0^2 / \gamma$ = time-integrated measurement variance.

$\zeta = \zeta_0 \gamma$ = feedback-shift rate.
Macroscopic view – Dynamics

Behaviour of centre of mass and total momentum can be easily obtained from the Master equation:

- No need for considering N-atom problem.
- Standard calculation for a single system with total mass \( M = N m \).

In case of harmonically trapped atoms (stationary solution):

- Average centre-of-mass and total momentum vanish
- Variances become:

\[
\begin{align*}
(\Delta X)^2 &\rightarrow (\Delta X_0)^2 (\eta + \eta^{-1})/2 \\
(\Delta P)^2 &\rightarrow (\Delta P_0)^2 \left[ \eta \left( 1 + (\zeta / \omega_0)^2 \right) + \eta^{-1} \right]/2
\end{align*}
\]

\[
\eta = (\Delta X_0)^2 / (\zeta \sigma^2) = \text{trap localisation / feedback localisation.}
\]

\( \omega_0 = \text{trap frequency.} \)
Problem – Many-atom correlations

Quantum measurement of collective centre of mass:

- Creates (quantum) correlations between atoms.
- Single-atom description only with approximations (typical many-particle problem).

True interactions between atoms also create correlations!

- Consider an ideal gas for simplicity,
- Pure effects due to feedback.

It is shown here, that: [Ivanov & Wallentowitz, Phys. Rev. Lett. (in press)]

- Single-atom motion is analytically solvable via trick,
- Motion is fed by noise,
- Noise is coloured due to atomic correlations.

Single-atom motion is non-Markovian!
Indistinguishable atoms

- Use second-quantized **bosonic matter field**:

\[ [\hat{\phi}(x), \hat{\phi}^\dagger(x')] = \delta(x - x') \]

**Relation between macroscopic and microscopic description:**

\[
\hat{X} = \frac{1}{\langle \hat{N} \rangle} \int dx \hat{\phi}^\dagger(x) x \hat{\phi}(x)
\]
\[
\hat{P} = -i\hbar \int dx \hat{\phi}^\dagger(x) \frac{\partial}{\partial x} \hat{\phi}(x)
\]

**Function of interest:** **reduced single-atom density matrix**

\[ \rho(x, x') = \langle \hat{\phi}^\dagger(x') \hat{\phi}(x) \rangle \]
Consider single atom in its surrounding (classical picture):

Quantum picture: + Symmetrisation of $N$-atom wavefunction
Two-body description – Wigner function

Wigner function for **single atom** + centre of mass of **N-1 “other” atoms**:  

\[
W_N(x, p; X, P) = (2\pi\hbar)^{-4} \int dx' \int dX' \int dP' \int dS \\
\times \exp\left\{ -i \left[ x'p + PX' + XP' + (N-1)S \right] / \hbar \right\} \\
\times \left\langle \phi^\dagger(x - x'/2) e^{i(PX' + XP' + NS) / \hbar} \phi(x + x'/2) \right\rangle.
\]

Properties of the Wigner function:

- Trace over \((X, P, N)\) gives single-atom Wigner function,
- Trace over \((x, p)\) gives Wigner function **only of quasi centre of mass**.
- **Single-atom dynamics is okay**, but subtlety with centre of mass!
- \((X, P) = \text{formal extension} \) of single-atom Hilbert space!
Two-body Wigner function: *Why and for what purpose?*

**Answer:**
- Wigner function obeys a closed Fokker-Planck equation.
- Is of **linear type** with positive **semi-definite** diffusion matrix.
- Gaussian Green's function & **bound analytic solutions!**

Instead of Fokker-Planck equation use **equivalent “Langevin” equations.**

**Advantages:**
- Identify **coupling** between microscopic and macroscopic systems.
- Properties of the driving **noise sources.**
Stochastic differential equations

Stochastic trajectory conditioned on having $N$ atoms in the system:

\[
\begin{align*}
\frac{dx_N}{dt} &= \left( \frac{p_N}{m} - \zeta \left( \frac{x_N}{\langle \hat{N} \rangle} + X_N \right) \right) dt + \eta \sigma \, d\xi_1, \\
\frac{dp_N}{dt} &= -m\omega_0^2 x_N \, dt + \frac{\hbar}{2\sigma \langle \hat{N} \rangle} \, d\xi_2, \\
\frac{dX_N}{dt} &= \left( \frac{P_N}{M} - \zeta \Theta_N \left( \frac{x_N}{\langle \hat{N} \rangle} + X_N \right) \right) dt + \Theta_N \eta \sigma \, d\xi_1, \\
\frac{dP_N}{dt} &= -M\omega_0^2 X_N \, dt + \frac{\hbar \Theta_N}{2\sigma} \, d\xi_2,
\end{align*}
\]

$M = m \langle \hat{N} \rangle = \text{total mass}$ \hspace{1cm} $\Theta_N = (N - 1)/\langle \hat{N} \rangle = \text{correction factor}$

$\xi_1, \xi_2 = \text{statistically independent Wiener processes}$
Elimination of macroscopic variables

**Decoupling in the limit of large atom number \( <N> \):**

Solve for \( X(t) \), \( P(t) \) and insert into \( dx(t) = ... + ... X(t) \ dt + ... \).

- Deterministic driving force.
- Addition to the noise source!

**Effective noise source for \( dx \):** = Wiener + Ornstein-Uhlenbeck process

\[
\begin{align*}
\frac{d\xi_N(t)}{dt} &= d\xi_1(t) - 2\Gamma_N \int_0^t \left\{ d\xi_1(t') \cos[\Omega_N(t-t')] - \frac{\eta d\xi_2(t') - \alpha_N d\xi_1(t')}{\sqrt{1-\alpha_N^2}} \sin[\Omega_N(t-t')] \right\} e^{-\Gamma_N(t-t')} dt' \\
\end{align*}
\]

Parameters: \( \Gamma_N = \zeta \Theta_N / 2 \), \( \Omega_N^2 = \omega_0^2 - \Gamma_N^2 \), \( \alpha_N = \Gamma_N / \omega_0 \)

**Single-atom dynamics is fed by coloured noise**

![](image.png)
Coloured noise spectrum

Fourier transform of stationary correlation function:

\[ S_N(\omega) = \lim_{t \to \infty} \int d\tau \ e^{i\omega \tau} \frac{\dot{\xi}_N(t+\tau)\dot{\xi}_N(t)}{\frac{1}{2} \frac{1}{2}} \]

Small feedback-induced damping: \( \alpha_N < 1/2^{1/2} \)
Explanation of noise reductions

Driven, damped harmonic oscillator:

*Destructive phase shift near resonance frequency!*
Measurable effects – Density profile

Asymptotic behaviour for large times: density oscillations

\[ \Delta x^2(t) \rightarrow \Delta X^2 + C(t) \]

Oscillating correlation function:

\[
C(t) = \langle \int \frac{dx}{N} \hat{\phi}^\dagger(x) [q(x, \partial_x, t)]^2 \hat{\phi}(x) \rangle \\
- \langle \int \frac{dx}{N} \hat{\phi}^\dagger(x) q(x, \partial_x, t) \hat{\phi}(x) \int \frac{dx'}{N} \hat{\phi}^\dagger(x') q(x', \partial_{x'}, t) \hat{\phi}(x') \rangle
\]

Freely oscillating quadrature:

\[ q(x, \partial_x, t) = x \cos(\omega_0 t) - \frac{i\hbar \partial_x}{m\omega_0} \sin(\omega_0 t) \]

Cauchy-Schwarz inequality: \textbf{Quantumness if} \boxed{C(t) < 1} \textbf{Observable as density oscillation below} \Delta X
Summary & conclusions:

**What has been shown:**

- **Analytic solution** of a correlated many-atom problem
- **Two-body** approach
- Single atom is fed by **coloured noise**
- **Non-Markovian** trajectories in phase space
- Observable via **density oscillations**

*Many-atom correlations have been cast into colour of noise feeding the single atom!*