PHYSICAL REVIEW A VOLUME 43, NUMBER 1 1 JANUARY 1991

# Minimum uncertainty states for amplitude-squared squeezing: Hermite polynomial states

János A. Bergou, Mark Hillery, and Daoqi Yu
*Department of Physics and Astronomy, Hunter College of the City University of New York, 695 Park Avenue, New York, New York 10021*
(Received 11 July 1990)

The real and imaginary parts of the square of the field amplitude are the variables that describe amplitude-squared squeezing. These quantities obey an uncertainty relation. Here we find a particularly simple subset of the states that satisfy the uncertainty relation as an equality. These states are constructed by applying a squeeze operator to a state that consists of a Hermite polynomial, whose argument is the mode creation operator multiplied by a constant, acting on the vacuum. The squeezed vacuum is such a state. These states may or may not be squeezed in the normal sense, and may or may not have sub-Poissonian photon statistics.

## I. INTRODUCTION

Squeezed states of the electromagnetic field are states for which the noise in a field quadrature component falls below its classically allowed value. In particular, for a single-mode field with creation and annihilation operators $a^\dagger$ and $a$, we define the quadrature components

$$X_1=(a^\dagger+a)/2,\quad X_2=i(a^\dagger-a)/2\ , \tag{1.1}$$

which obey the uncertainty relation ($\hbar=1$)

$$\Delta X_1\Delta X_2\geq\tfrac{1}{4}\ . \tag{1.2}$$

If $\Delta X_1<\frac{1}{2}$ for a particular state, then that state is squeezed in the $X_1$ direction. A classical state, i.e., one with a non-negative definite $P$ representation, must satisfy $\Delta X_1>\frac{1}{2}$ so that squeezed states are nonclassical.

The states that satisfy Eq. (1.2) as an equality are the minimum uncertainty states for the variables $X_1$ and $X_2$.[1] Such states can be generated from the vacuum by the application of an appropriate squeeze operator,

$$S(z)=\exp\{[z(a^\dagger)^2-z^*a^2]/2\}\ , \tag{1.3}$$

followed by a displacement operator,

$$D(\alpha)=\exp(\alpha a^\dagger-\alpha^*a)\ . \tag{1.4}$$

That is, $D(\alpha)S(z)|0\rangle$ is a minimum uncertainty state for $X_1$ and $X_2$ if $z$ is real or pure imaginary. If $z\neq0$, then the state is squeezed, and, in particular, if $z\neq0$ and $\alpha=0$, then the state is called a squeezed vacuum. If $z=0$, then the state is called a coherent state.

It is also possible to define squeezing for variables other than $X_1$ and $X_2$. One can, for example, consider squeezing in variables that are quadratic in the field operators. The specific variables that we wish to examine are the real and imaginary parts of the square of the field amplitude.[2-4] The operators corresponding to these quantities are

$$Y_1=[(a^\dagger)^2+a^2]/2,\quad Y_2=i[(a^\dagger)^2-a^2]/2\ . \tag{1.5}$$

The commutator of $Y_1$ and $Y_2$ is

$$[Y_1,Y_2]=i(2N+1)\ , \tag{1.6}$$

where $N=a^\dagger a$ is the number operator. As a result, they obey the uncertainty relation:

$$\Delta Y_1\Delta Y_2\geq\langle N+\tfrac{1}{2}\rangle\ . \tag{1.7}$$

A state is said to be amplitude-squared squeezed in the $Y_1$ direction if $(\Delta Y_1)^2<\langle N+\frac{1}{2}\rangle$. Such states are nonclassical. Amplitude-squared squeezing can be converted into normal squeezing by second-harmonic generation.[2]

Two different kinds of states have been studied in connection with amplitude-squared squeezing. The first are the SU(1,1) coherent states. These are natural to consider because the operators $a^2/2,(a^\dagger)^2/2$, and $(a^\dagger a+aa^\dagger)/4$ form a representation of the su(1,1) Lie algebra. Amplitude-squared squeezing was, in fact, first discussed by Wodkiewicz and Eberly in connection with these states.[4] SU(1,1) coherent states can exhibit amplitude-squared squeezing but do not, in general, satisfy Eq. (1.7) as an equality. They are produced from the vacuum by the action of a degenerate parametric amplifier.[4,5] The second class of states consists of the even and odd coherent states.[6] These are both eigenstates of $a^2$ (but not of $a$) and do satisfy Eq. (1.7) as an equality.[3] They are not, however, amplitude-squared squeezed. These states can also be considered within the context of the su(1,1) Lie algebra, and this has been done by Bužek.[7]

The object of this paper is to find a subset of the minimum uncertainty states for amplitude-squared squeezing. This subset has been chosen for its simplicity, and we shall present the somewhat more complicated general case in a subsequent publication. What we mean by a minimum uncertainty state is a state for which Eq. (1.7) is satisfied as an equality. These states have been called "intelligent states" by some authors,[4,8] but we shall retain what we believe to be the more conventional nomenclature.

The paper is organized as follows. In Sec. II we discuss the eigenvalue equation that allows one to find the

 ©1991 The American Physical Society

minimum uncertainty states and some of the implications it has for their properties. In Sec. III we solve the equation and present the subset of its solutions which has a particularly simple form. In Sec. IV the properties of these states are discussed. Our results are summarized in Sec. V.

## II. EIGENVALUE EQUATION

The minimum uncertainty states for amplitude-squared squeezing are the solutions to the eigenvalue problem:

$$(Y_1+i\lambda Y_2)|\psi\rangle=\beta|\psi\rangle \; , \tag{2.1}$$

where $\lambda$ is real, and $\beta$ is complex. This follows from examining the difference between the two sides in Eq. (1.7) and demanding that it be zero. The general procedure for accomplishing this is discussed in detail in Ref. 9, and it implies that only the states that satisfy Eq. (2.1) will satisfy Eq. (1.7) as an equality. We shall prove, in a somewhat different fashion, that the states that satisfy Eq. (2.1) are minimum uncertainty states. In doing so we gain insight into the meaning of the parameters $\lambda$ and $\beta$. In Sec. III we shall solve this equation in order to find the minimum uncertainty states.

As a first step, it is useful to take the expectation of the operator $Y_1+i\lambda Y_2$ in the state $|\psi\rangle$ that is a solution of Eq. (2.1). This gives

$$\langle\psi|Y_1|\psi\rangle+i\lambda\langle\psi|Y_2|\psi\rangle=\beta \; . \tag{2.2}$$

Because $Y_1$ and $Y_2$ are Hermitian, their expectation values are real. Thus Eq. (2.2) implies that

$$\langle\psi|Y_1|\psi\rangle=\beta_r, \quad \langle\psi|Y_2|\psi\rangle=\beta_i/\lambda \; , \tag{2.3}$$

where $\beta_r=\mathrm{Re}\beta$ and $\beta_i=\mathrm{Im}\beta$. Therefore $\beta$ is directly related to the expectation values of $Y_1$ and $Y_2$ in minimum uncertainty states.

Now let us multiply Eq. (2.1) by the operator $Y_1-i\lambda Y_2$ and then take the inner product with $|\psi\rangle$. The result, after making use of Eq. (2.3), is

$$\langle\psi|Y_1^2+\lambda^2Y_2^2|\psi\rangle+i\lambda\langle\psi|[Y_1,Y_2]|\psi\rangle=|\beta|^2 \; ,$$

or

$$(\Delta Y_1)^2+\lambda^2(\Delta Y_2)^2=\lambda\langle\psi|2N+1|\psi\rangle \; . \tag{2.4}$$

Note that this equation implies that $\lambda$ must be greater than or equal to zero.

Multiplying Eq. (2.1) by $Y_1+i\lambda Y_2$ and taking the inner product with $|\psi\rangle$ also provides useful information. After doing so, and then taking the real and imaginary parts, we find that

$$\begin{aligned}&\langle\psi|Y_1^2-\lambda^2Y_2^2|\psi\rangle=\mathrm{Re}(\beta^2) \; ,\\ &\lambda\langle\psi|Y_1Y_2+Y_2Y_1|\psi\rangle=\mathrm{Im}(\beta^2) \; .\end{aligned} \tag{2.5}$$

If we now combine the first of these equations with the results in Eq. (2.2), we have

$$(\Delta Y_1)^2-\lambda^2(\Delta Y_2)^2=\mathrm{Re}(\beta^2)-\beta_r^2+\beta_i^2=0 \; ,$$

or

$$\Delta Y_1=\lambda\Delta Y_2 \; . \tag{2.6}$$

From this equation, it is clear that $\lambda$ plays the role of a squeezing parameter. If $\lambda=1$, then the uncertainties are equal, and there is no amplitude-squared squeezing. If $\lambda>1$, then $\Delta Y_2$ is squeezed, and if $0<\lambda<1$, then $\Delta Y_1$ is squeezed.

Finally, if Eq. (2.6) is used to eliminate either $\Delta Y_1$ or $\Delta Y_2$ in Eq. (2.4), then we find that

$$\begin{aligned}&(\Delta Y_1)^2=\lambda\langle\psi|N+\tfrac{1}{2}|\psi\rangle \; ,\\ &(\Delta Y_2)^2=(1/\lambda)\langle\psi|N+\tfrac{1}{2}|\psi\rangle \; .\end{aligned} \tag{2.7}$$

These equations show that $\Delta Y_1\Delta Y_2=\langle\psi|N+\frac{1}{2}|\psi\rangle$ and prove that a solution of Eq. (2.1) is a minimum uncertainty state. They confirm, as well, the comments made in the preceding paragraph about the role of $\lambda$ as a squeezing parameter. Note, also, that if we know that $|\psi\rangle$ is a solution of Eq. (2.1), then the photon number determines $\Delta Y_1$ and $\Delta Y_2$. That is, if $\langle\psi|N|\psi\rangle$ is known, then Eqs. (2.7) can be used to find $\Delta Y_1$ and $\Delta Y_2$ immediately, thus providing a certain economy of computation. This fact will prove useful later.

## III. SOLUTION OF EIGENVALUE EQUATION

We now want to solve Eq. (2.1) for $|\psi\rangle$. It is first useful to express the equation in terms of creation and annihilation operators:

$$[(1-\lambda)(a^\dagger)^2/2+(1+\lambda)a^2/2]|\psi\rangle=\beta|\psi\rangle \; . \tag{3.1}$$

A natural approach, at this point, is to expand $|\psi\rangle$ in terms of number states. This, however, leads to a three-term recurrence relation for the expansion coefficients, which is difficult to solve. It is instead simpler to introduce the state

$$|\psi'\rangle=S(z)^{-1}|\psi\rangle \; , \tag{3.2}$$

which is related to $|\psi\rangle$ by the squeezing transformation $S(z)$. The parameter $z$ will be chosen later. The state $|\psi'\rangle$ can then be expanded in photon number states. For the proper choice of $z$, the recurrence relation that determines the expansion coefficients contains only two terms and can be easily solved.

If we let $z=re^{i\theta}$, then we find that $|\psi'\rangle$ satisfies

$$S(z)^{-1}[(1-\lambda)(a^\dagger)^2/2+(1+\lambda)a^2/2]S(z)|\psi'\rangle=\beta|\psi'\rangle \; ,$$

or

$$\{[(1-\lambda)(\cosh^2 r)/2+(1+\lambda)(e^{2i\theta}\sinh^2 r)/2](a^\dagger)^2+[(1-\lambda)e^{-i\theta}/2+(1+\lambda)e^{i\theta}/2]\cosh r\,\sinh r(a^\dagger a+aa^\dagger)$$
$$+[(1-\lambda)(e^{-2i\theta}\sinh^2 r)/2+(1+\lambda)\cosh^2 r/2]a^2\}|\psi'\rangle=\beta|\psi'\rangle\ . \quad (3.3)$$

We now want to choose $z$, so that the coefficient of $(a^\dagger)^2$ vanishes. The condition for this is

$$\tanh^2 r=e^{-2i\theta}(\lambda-1)/(\lambda+1)\ . \quad (3.4)$$

For $\lambda\geq 1$ we choose $\theta=0$, and for $0<\lambda<1$ we take $\theta=\pi/2$. The parameter $r$ is then chosen, so that $\tanh^2 r$ is equal to the absolute value of the right-hand side. Note that this is always possible, as $|(\lambda-1)/(\lambda+1)|<1$ for $\lambda>0$. With these choices we find that, for $0<\lambda<1$,

$$\cosh r=[(1+\lambda)/2\lambda]^{1/2}\ ,\qquad \sinh r=[(1-\lambda)/2\lambda]^{1/2}\ , \quad (3.5)$$

and for $\lambda\geq 1$,

$$\cosh r=[(1+\lambda)/2]^{1/2}\ ,\qquad \sinh r=[(\lambda-1)/2]^{1/2}\ . \quad (3.6)$$

These expressions can now be substituted into Eq. (3.3). The result for $0<\lambda<1$ is

$$[a^2+i(1-\lambda^2)^{1/2}(a^\dagger a+\tfrac{1}{2})]|\psi'\rangle=\beta|\psi'\rangle\ , \quad (3.7)$$

and for $\lambda\geq 1$ it is

$$[\lambda a^2+(\lambda^2-1)^{1/2}(a^\dagger a+\tfrac{1}{2})]|\psi'\rangle=\beta|\psi'\rangle\ . \quad (3.8)$$

We now expand $|\psi'\rangle$ in terms of number states

$$|\psi'\rangle=\sum_{n=0}^{\infty}c_n|n\rangle\ , \quad (3.9)$$

and substitute this expression into Eqs. (3.7) and (3.8). This leads to the recurrence relations

$$c_{n+2}=\left[\frac{\beta-i(1-\lambda^2)^{1/2}(n+\frac{1}{2})}{[(n+1)(n+2)]^{1/2}}\right]c_n \quad (3.10)$$

for $0<\lambda<1$, and

$$c_{n+2}=\left[\frac{\beta-(\lambda^2-1)^{1/2}(n+\frac{1}{2})}{\lambda[(n+1)(n+2)]^{1/2}}\right]c_n \quad (3.11)$$

for $\lambda\geq 1$. Both $c_0$ and $c_1$ are arbitrary.

These recurrence relations are easily solved for any value of $\beta$ and for $\lambda>0$. The properties of the states that result from the general solution will be discussed in a subsequent paper. Here, we wish to examine a particularly simple subset of solutions. This subset is found by noting that, if $\beta$ and $\lambda$ are related in the proper way, only a finite number of the coefficients $c_n$ will be different from zero. In particular, if $0<\lambda<1$ and $\beta=i(1-\lambda^2)^{1/2}(m+\frac{1}{2})$, where $m$ is a non-negative integer, then the series for $|\psi'\rangle$ can be chosen to terminate with $c_m|m\rangle$ being the last term. Similarly, if $\lambda\geq 1$, and $\beta=(\lambda^2-1)^{1/2}(m+\frac{1}{2})$, again with $m$ as a non-negative integer, then $c_0$ and $c_1$ can be chosen, so that $c_n=0$ for $n>m$.

Let us find explicit expressions for the solutions. If $0<\lambda<1$, then for even $m$ we have

$$|\psi'(m,\lambda)\rangle=\sum_{n=0}^{m/2}[i(1-\lambda^2)^{1/2}]^n\times\frac{2^n}{\sqrt{(2n)!}}\frac{(m/2)!}{(m/2-n)!}c_0|2n\rangle\ , \quad (3.12)$$

and for odd $m$ we have

$$|\psi'(m,\lambda)\rangle=\sum_{n=0}^{(m-1)/2}[i(1-\lambda^2)^{1/2}]^n\times\frac{2^n}{\sqrt{(2n+1)!}}\frac{[(m-1)/2]!}{[(m-1)/2-n]!}\times c_1|2n+1\rangle\ . \quad (3.13)$$

If $\lambda\geq 1$, then for even $m$ we have

$$|\psi'(m,\lambda)\rangle=\sum_{n=0}^{m/2}[(\lambda^2-1)^{1/2}/\lambda]^n\times\frac{2^n}{\sqrt{(2n)!}}\frac{(m/2)!}{(m/2-n)!}c_0|2n\rangle\ , \quad (3.14)$$

and for odd $m$ we have

$$|\psi'(m,\lambda)\rangle=\sum_{n=0}^{(m-1)/2}[(\lambda^2-1)^{1/2}/\lambda]^n\times\frac{2^n}{\sqrt{(2n+1)!}}\frac{[(m-1)/2]!}{[(m-1)/2-n]!}\times c_1|2n+1\rangle\ . \quad (3.15)$$

The constants $c_0$ and $c_1$ are chosen so as to normalize the states.

It is possible to express these states in a relatively compact form in terms of Hermite polynomials. The $n$th Hermite polynomial is given by

$$H_n(x)=\sum_{k=0}^{[n/2]}(-1)^k(2x)^{n-2k}n!/[(n-2k)!k!]\ , \quad (3.16)$$

where $[n/2]$ denotes the greatest integer less than or equal to $n/2$. The state $|\psi'(m,\lambda)\rangle$ can be expressed as

$$|\psi'(m,\lambda)\rangle=c_m(\lambda)H_m(i\gamma(\lambda)a^\dagger)|0\rangle\ , \quad (3.17)$$

where $c_m(\lambda)$ is a normalization constant, and

$$\gamma(\lambda)=e^{i\pi/4}[(1-\lambda^2)^{1/2}/2]^{1/2} \quad (3.18)$$

for $0<\lambda<1$, and

$$\gamma(\lambda)=[(\lambda^2-1)^{1/2}/2\lambda]^{1/2} \quad (3.19)$$

for $\lambda\geq 1$. Finally, we can combine this with the squeezing transformation to give the minimum uncertainty states:

$$|\psi(m,\lambda)\rangle = S(z)|\psi'(m,\lambda)\rangle = c_m(\lambda)S(z)H_m(i\gamma(\lambda)a^\dagger)|0\rangle \ , \quad (3.20)$$

where $z$ is defined by Eq. (3.4). It is these states that we now wish to examine further. It should again be noted that the states $|\psi(m,\lambda)\rangle$ are only a subset of the full set of minimum uncertainty states.

## IV. PROPERTIES OF STATES

We now want to discuss some of the properties of the states $|\psi(m,\lambda)\rangle$. First, we shall obtain more explicit expressions for the photon number and for the uncertainties in $Y_1$ and $Y_2$. Then, it will be determined whether these states are squeezed in the normal sense and whether their photon statistics are sub-Poissonian.

Equation (2.7) shows that if we know the average photon number for these states, then we know $\Delta Y_1$ and $\Delta Y_2$. Therefore we need to find $\langle \psi(m,\lambda)|N|\psi(m,\lambda)\rangle$. This quantity can, in fact, be expressed rather simply in terms of the functions $|c_m(\lambda)|^2$. In order to see this, first note that

$$\langle \psi(m,\lambda)|N|\psi(m,\lambda)\rangle = |c_m(\lambda)|^2\{(\cosh^2 r+\sinh^2 r)\langle 0|H_m(-i\gamma^* a)a^\dagger a H_m(i\gamma a^\dagger)|0\rangle + \cosh r \sinh r[e^{-i\theta}\langle 0|H_m(-i\gamma^* a)a^2 H_m(i\gamma a^\dagger)|0\rangle + e^{i\theta}\langle 0|H_m(-i\gamma^* a)(a^\dagger)^2 H_m(i\gamma a^\dagger)|0\rangle]\} + \sinh^2 r \ . \quad (4.1)$$

The equation

$$\langle \psi(m,\lambda)|a^2|\psi(m,\lambda)\rangle = \beta_r + i\beta_i/\lambda \ , \quad (4.2)$$

which follows from Eq. (2.3), and its complex conjugate allow us to solve for $\langle 0|H_m(-i\gamma^* a)a^2 H_m(i\gamma a^\dagger)|0\rangle$ and $\langle 0|H_m(-i\gamma^* a)(a^\dagger)^2 H_m(i\gamma a^\dagger)|0\rangle$ in terms of $\langle 0|H_m(-i\gamma^* a)a^\dagger a H_m(i\gamma a^\dagger)|0\rangle$, $\beta_r$, and $\beta_i/\lambda$. The result for the expectation value of $a^2$ is

$$(\cosh^2 r+\sinh^2 r)\langle 0|H_m(-i\gamma^* a)a^2 H_m(i\gamma a^\dagger)|0\rangle = [(\cosh^2 r - e^{2i\theta}\sinh^2 r)\beta_r + i(\cosh^2 r + e^{2i\theta}\sinh^2 r)\beta_i/\lambda]/|c_m(\lambda)|^2 - e^{i\theta}\cosh r \sinh r\langle 0|H_m(-i\gamma^* a)(2a^\dagger a+1)H_m(i\gamma a^\dagger)|0\rangle \ . \quad (4.3)$$

This result and its complex conjugate can then be substituted into Eq. (4.1). It remains to simplify the expectation value of $a^\dagger a$ in the Hermite polynomial state. Here, the identity

$$aH_m(i\gamma a^\dagger)|0\rangle = 2i\gamma m H_{m-1}(i\gamma a^\dagger)|0\rangle \quad (4.4)$$

leads to

$$\langle 0|H_m(-i\gamma^* a)a^\dagger a H_m(i\gamma a^\dagger)|0\rangle = 4|\gamma|^2 m^2\langle 0|H_{m-1}(-i\gamma^* a)H_{m-1}(i\gamma a^\dagger)|0\rangle = 4m^2|\gamma|^2/|c_{m-1}(\lambda)|^2 \ . \quad (4.5)$$

Finally, expressing the quantities $\beta$, $\gamma$, $\cosh r$, and $\sinh r$ in terms of $\lambda$ and $m$, for $0<\lambda<1$ we get

$$\langle \psi(m,\lambda)|N|\psi(m,\lambda)\rangle = 2m^2\lambda(1-\lambda^2)^{1/2}|c_m(\lambda)|^2/|c_{m-1}(\lambda)|^2 + (1-\lambda^2)(m+\tfrac{1}{2})/\lambda - (1-\lambda)/2 \ , \quad (4.6)$$

and for $\lambda \geq 1$ we get

$$\langle \psi(m,\lambda)|N|\psi(m,\lambda)\rangle = 2m^2(\lambda^2-1)^{1/2}|c_m(\lambda)|^2/\lambda^2|c_{m-1}(\lambda)|^2 + (\lambda^2-1)(m+\tfrac{1}{2})/\lambda - (\lambda-1)/2\lambda \ . \quad (4.7)$$

Let us now use these expressions to find $\langle N\rangle$, $\Delta Y_1$, and $\Delta Y_2$ for $|\psi(m,\lambda)\rangle$, for the cases $m=0$, 1, and 2. A straightforward calculation gives

$$|c_0(\lambda)|^2 = 1, \quad |c_1(\lambda)|^2 = 1/4|\gamma|^2 \ , \quad |c_2(\lambda)|^2 = 1/(32|\gamma|^4+4) \ . \quad (4.8)$$

Turning first to $|\psi(1,\lambda)\rangle$, which is a squeezed one-photon state, we find for $0<\lambda<1$ that

$$\langle N\rangle = (3-\lambda)/2\lambda \ , \quad (\Delta Y_1)^2 = \tfrac{3}{2} \ , \quad (\Delta Y_2)^2 = 3/2\lambda^2 \ , \quad (4.9)$$

and for $\lambda \geq 1$ that

$$\langle N\rangle = (3\lambda-1)/2 \ , \quad (\Delta Y_1)^2 = 3\lambda^2/2 \ , \quad (\Delta Y_2)^2 = \tfrac{3}{2} \ . \quad (4.10)$$

Note that $\langle N\rangle \to \infty$ as $\lambda$ goes to either zero or infinity. As $\lambda\to 0$, $Y_1$ becomes increasingly squeezed, and $(\Delta Y_1)^2$ is equal to $\frac{3}{2}$. Similarly, as $\lambda\to\infty$, $Y_2$ becomes more and more squeezed, and $(\Delta Y_2)^2$ is equal to $\frac{3}{2}$.

The state $|\psi(2,\lambda)\rangle$ is obtained by squeezing a linear combination of the vacuum and a two-photon state. For this state, the photon number is

$$\langle N\rangle = \begin{cases} [(15-14\lambda^2)/(6-4\lambda^2)\lambda] - \frac{1}{2}, & 0<\lambda<1 \\ [(15\lambda^3-14\lambda)/(6\lambda^2-4)] - \frac{1}{2}, & \lambda \geq 1 \ . \end{cases} \quad (4.11)$$

This again diverges as $\lambda$ goes to either zero or infinity. In this case, however, as $\lambda$ goes to zero, $(\Delta Y_1)^2$ goes to $\frac{5}{2}$, and as $\lambda$ goes to infinity, $(\Delta Y_2)^2$ goes to $\frac{5}{2}$.

It is also possible to derive an asymptotic expression for $|c_m(\lambda)|^2$ which is valid for large $m$. This is done in the Appendix. If the result is used to find the mean photon number for $|\psi(m,\lambda)\rangle$, then we find for large $m$ that

$$\langle N\rangle = \begin{cases} (m+\frac{1}{2})/\lambda - m\lambda/[1+(1-\lambda^2)^{1/2}], & 0<\lambda<1 \\ \lambda(m+\frac{1}{2}) - m/[\lambda+(\lambda^2-1)^{1/2}], & \lambda\geq 1\ . \end{cases} \tag{4.12}$$

Therefore, for large $m$ we have a situation similar to that for $m=1$ and 2. That is, as $\lambda\to 0$, the photon number diverges and $(\Delta Y_1)^2\to(m+\frac{1}{2})$. As $\lambda\to\infty$, the photon number also diverges and $(\Delta Y_2)^2\to(m+\frac{1}{2})$.

The simplest case is $m=0$. The state $|\psi(0,\lambda)\rangle$ is just a squeezed vacuum, so that this state is a minimum uncertainty state for both normal squeezing and amplitude-squared squeezing. The photon number, in this case, is

$$\langle N\rangle = \begin{cases} (1/2\lambda)-\frac{1}{2}, & 0<\lambda<1 \\ (\lambda-1)/2, & \lambda\geq 1\ , \end{cases} \tag{4.13}$$

and we can see that this state follows the pattern set by those which we have discussed previously.

It is of interest to determine whether the states $|\psi(m,\lambda)\rangle$ are squeezed in the normal sense. Clearly, $|\psi(0,\lambda)\rangle$ being a squeezed vacuum is squeezed for all $\lambda\neq 1$. This is, however, not the case for $m\neq 0$. A short calculation shows that the state $|\psi(1,\lambda)\rangle$ is not squeezed if $\frac{3}{5}<\lambda<\frac{5}{3}$, but is squeezed otherwise. Therefore, amplitude-squared squeezed minimum uncertainty states may or may not be squeezed in the normal sense.

A similar conclusion is reached if we inquire whether these states have sub-Poissonian photon statistics. Here, we start by deriving a simple expression for $(\Delta N)^2$ for $|\psi(m,\lambda)\rangle$. Noting that $Y_1^2+Y_2^2=N^2+N+1$, we obtain

$$(\Delta Y_1)^2+(\Delta Y_2)^2=\langle N^2+N+1\rangle-\beta_r^2-(\beta_i/\lambda)^2\ . \tag{4.14}$$

Using Eq. (2.7) and solving the resulting expression for $(\Delta N)^2-\langle N\rangle$, we get

$$(\Delta N)^2-\langle N\rangle=[(\lambda^2+1)/\lambda]\langle N+\tfrac{1}{2}\rangle +\beta_r^2+(\beta_i/\lambda)^2-\langle N+1\rangle^2\ . \tag{4.15}$$

If the right-hand side of Eq. (4.15) is negative, the photon statistics of $|\psi(m,\lambda)$ are sub-Poissonian. To illustrate what can happen, let us consider the case of $m=1$. Because $|\psi(1,1)\rangle$ is just the one-photon number state, it is clear that for some range of $\lambda$ the photon statistics are sub-Poissonian. A more detailed analysis, using Eqs. (4.9), (4.10), and (4.15), shows that they are sub-Poissonian if

$$2/[1+(\tfrac{11}{3})^{1/2}]<\lambda<[1+(\tfrac{11}{3})^{1/2}]/2\ , \tag{4.16}$$

and they are not if $\lambda$ is outside this range. One expects that this behavior is typical for odd $m$. The state $|\psi(m,1)\rangle$, for odd $m$, is just the one-photon number state and is, therefore, sub-Poissonian. As $\lambda$ deviates from 1, the amount of squeezing in the state increases and eventually makes the state super-Poissonian.

## V. CONCLUSION

We have presented a subset of the minimum uncertainty states $|\psi(m,\lambda)\rangle$ for the variables that describe amplitude-squared squeezing. For these states, the expectation values of $Y_1$ and $Y_2$ are related to their fluctuations. This is not true for a general amplitude-squared minimum uncertainty state, where the expectation values of $Y_1$ and $Y_2$ can be specified independently of the squeezing parameter $\lambda$. The complete set of minimum uncertainty states will be discussed in a future publication.

All of the states $|\psi(m,\lambda)\rangle$ have the property that they have zero mean amplitude, i.e., the expectation of the annihilation operator is zero. If one displaces one of these states with a displacement operator $D(\alpha)$, the minimum uncertainty character of the state is destroyed. This is because the amount of amplitude-squared squeezing, unlike the amount of normal squeezing, is not invariant under displacements.

Perhaps our most interesting result is that the squeezed vacuum is a minimum uncertainty state for amplitude-squared squeezing, as well as for normal squeezing. Because such a state can be produced by a degenerate parametric amplifier, it is the most likely of the states $|\psi(m,\lambda)\rangle$ to be seen in the laboratory. The other candidate for a state, which is, perhaps, not too hard to observe, is $|\psi(1,\lambda)\rangle$. This is a squeezed one-photon state and can be produced by using a one-photon number state as the input for a degenerate parametric amplifier.

Finally, we note that as the amount of amplitude-squared squeezing in these states increases, so does the photon number. A similar situation occurs with normal squeezing.[10] Thus amplitude-squared squeezed minimum uncertainty states provide another example of states whose nonclassical behavior becomes most pronounced at large photon number.

## ACKNOWLEDGMENTS

This research was supported by a grant from the City University of New York PSC-CUNY Research Award Program and by the National Science Foundation under Grant No. PHY-8802683.

## APPENDIX

Here we want to find an asymptotic expression for $|c_n(\lambda)|^2$ that is valid as $n\to\infty$. We start by noting the expression for the generating function for the Hermite polynomials:

$$e^{2zx-z^2}=\sum_{n=0}^{\infty}(z^n/n!)H_n(x)\ . \tag{A1}$$

Application of this result twice gives

$$e^{-[z^2+(z^*)^2]}\langle 0|e^{-2iz^*\gamma^* a}e^{2iz\gamma a^\dagger}|0\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[(z^*)^n z^m]/(n!m!) \times\langle 0|H_n(-i\gamma^* a)H_m(i\gamma a^\dagger)|0\rangle . \quad (A2)$$

The left-hand side of this equation can be simplified by noting that

$$e^{4|z|^2|\gamma|^2}=\langle 0|e^{-2iz^*\gamma^* a}e^{2iz\gamma a^\dagger}|0\rangle . \quad (A3)$$

Now, set $z=re^{i\theta}$ and integrate both sides of Eq. (A2), with respect to $\theta$, from 0 to $2\pi$. The result is

$$(1/2\pi)e^{4r^2|\gamma|^2}\int_0^{2\pi}d\theta e^{-2r^2\cos(2\theta)} = \sum_{n=0}^{\infty}[(r^2)^n/(n!)^2]\langle 0|H_n(-i\gamma^* a)H_n(i\gamma a^\dagger)|0\rangle . \quad (A4)$$

Equating the powers of $r^2$ on both sides, we get

$$\langle 0|H_n(-i\gamma^* a)H_n(i\gamma a^\dagger)|0\rangle =[(2^{n-1}n!)/\pi]\int_0^{2\pi}d\theta[2|\gamma|^2-\cos(2\theta)]^n =[(2^{n-1}n!)/\pi]\int_0^{2\pi}d\theta(2|\gamma|^2-\cos\theta)^n . \quad (A5)$$

In order to find an asymptotic form for the integral as $n\to\infty$, we first express the integrand as $\exp[n\ln(2|\gamma|^2-\cos\theta)]$ and apply the Laplace method.[11] This involves finding the maximum of $\ln(2|\gamma|^2-\cos\theta)$ (which occurs at $\theta=\pi$) and then expanding about it, keeping only lowest-order terms. The resulting integral is Gaussian and can be performed. The result is

$$\int_0^{2\pi}d\theta(2|\gamma|^2-\cos\theta)^n \cong(2|\gamma|^2+1)^n[\pi(4|\gamma|^2+2)/n]^{1/2} , \quad (A6)$$

so that

$$\langle 0|H_n(-i\gamma^* a)H_n(i\gamma a^\dagger)|0\rangle \cong 2^{n-1}n!(2|\gamma|^2+1)^n[(4|\gamma|^2+2)/n\pi]^{1/2} \quad (A7)$$

for large $n$. The function $|c_n(\lambda)|^2$ is simply the reciprocal of this expression.

[1] D. Stoler, Phys. Rev. D **4**, 2309 (1971).
[2] M. Hillery, Opt. Commun. **62**, 135 (1987).
[3] M. Hillery, Phys. Rev. A **36**, 3796 (1987).
[4] K. Wodkiewicz and J. H. Eberly, J. Opt. Soc. B **2**, 458 (1985).
[5] C. C. Gerry, Phys. Rev. A **31**, 2721 (1985).
[6] J. Perina, *Quantum Statistics of Linear and Nonlinear Optical Phenomena* (Reidel, Dordrecht, 1984), p. 78.
[7] V. Bužek, J. Mod. Opt. **37**, 303 (1990).
[8] C. Aragone, G. Guerri, and S. Salamo, J. Math. Phys. **17**, 1963 (1976).
[9] L. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968), p. 61.
[10] M. Hillery, Phys. Rev. A **39**, 1556 (1989).
[11] A. Erdelyi, *Asymptotic Expansions* (Dover, New York, 1956), p. 36.