On intelligent spin states

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(Received 14 July 1975; revised manuscript received 9 April 1976)

In this paper we give a more compact representation of the intelligent spin states defined by Aragone, Guerri, Salamó, and Tani. Using this new representation, we discuss the differences between minimum uncertainty states, coherent Bloch spin states and intelligent states. The evolution of these states under a particular time dependent Hamiltonian is studied, showing the relevance of the noncompact subgroup K of the Lorentz group. Finally we analyze the radiative properties connected with the intelligent states for a pointlike medium. The main results are: (I) they have a nonvanishing dipole moment (as the Bloch states) and (II) the proper intelligent states give a spontaneous emission intensity which is different from the one provided by the Bloch states.

1. INTRODUCTION

In a recent paper, Aragone, Guerri, Salamó and Tani,[1] constructed the intelligent spin states as those which satisfy the Heisenberg equality for the angular momentum operators. Many questions of physical interest were not discussed there.

The purpose of this work is threefold: (a) to give a clear distinction between intelligent states, minimum uncertainty states, and Bloch states; (b) to show a more compact representation of intelligent states; and (c) to determine the time evolution and some radiative properties of two different systems initially set in an intelligent state.

This article is organized as follows: In the next section we give a more compact expression for the intelligent states than the original, and we discuss the connection between intelligent states and coherent spin states.[2-4] We will show the difference between the $2j+1$ intelligent states and the $2j+1$ states obtained by applying the two-parameter rotation $R(\tau)$, defined by Arecchi, Courtens, Gilmore, and Thomas (ACGT),[4] to the standard Wigner states $|j,m\rangle$.

Section 3 is devoted to analyzing the difference between minimum uncertainty states, atomic coherent spin states, and intelligent states. We calculate the expectation values of J_x, J_y, J_z and their quadratic deviations for intelligent states, using the technique of generating functionals, whose details are presented in Appendix A.

In Sec. 4 we present the explicit evolution of a nonrelativistic high spin system, initially set in an intelligent state, immersed in a magnetic atmosphere.

We also estimate the macroscopic dipole and emission rates of a pointlike laser.

In the last section we make some comments and remarks.

2. COHERENT SPIN STATES AND INTELLIGENT STATES

The SU(2) algebra is defined by the usual commutation relations,

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad i,j,k=1,2,3, \tag{1a}$$

or, in terms of the ladder operators $J_\epsilon \equiv J_1 + i\epsilon J_2$ ($\epsilon = +1, -1$) and J_3, by

$$[J_\epsilon, J_{-\epsilon}] = 2\epsilon J_3, \quad [J_3, J_\epsilon] = \epsilon J_\epsilon. \tag{1b}$$

The $(2j+1)$-dimensional Hilbert space spanned by the eigenvectors of J^2 and J_3 (labeled by $|j,m\rangle$ or by $|m\rangle$)

$$J^2|j,m\rangle = j(j+1)|j,m\rangle, \quad J_3|j,m\rangle = m|j,m\rangle, \tag{2}$$

is denoted by H_j.

A useful formula for computation is

$$(j+\epsilon m)!\,|m\rangle = \binom{2j}{j+\epsilon m}^{-1/2} J_\epsilon^{j+\epsilon m}|-\epsilon j\rangle. \tag{3}$$

The ladder operators are useful in order to construct[4] the atomic coherent spin states or Bloch states $|\tau\rangle$,

$$\begin{aligned}|\tau\rangle &\equiv (1+|\tau|^2)^{-j}\exp(\tau J_+)|-j\rangle \\ &= \exp(\tau J_+)\exp[\ln(1+|\tau|^2)J_3]\exp(-\tau^* J_-|-j\rangle \\ &\equiv R(\tau)|-j\rangle,\end{aligned} \tag{4}$$

where $\tau \equiv \tan\frac{1}{2}\theta\exp(-i\varphi)$, $\theta \in [0, 2\pi)$. $R(\tau)$ represents a rotation through an angle θ about the axis $\hat{a} \equiv \sin\varphi\,\hat{e}_1 - \cos\varphi\,\hat{e}_2$.

Two different Bloch states are not necessarily orthogonal. In fact their inner product is

$$\langle\tau_1|\tau_2\rangle = (1+|\tau_1|^2)^{-j}(1+|\tau_2|^2)^{-j}(1+\tau_1^*\tau_2)^{2j}. \tag{5}$$

The expression of the atomic coherent spin given in Eq. (4) is analogous to that for Glauber states, $|z\rangle = N(z)\exp(za+)|0\rangle$, where the operator $\exp(za^+)$ is applied to the ground state of the harmonic oscillator.[5,6]

The Glauber states satisfy the Heisenberg equality $\Delta x\,\Delta p = \frac{1}{2}$. Therefore, one could also enquire whether the states $|\tau\rangle$ satisfy the Heisenberg equality for the SU(2) algebra,

$$\Delta J_1^2\,\Delta J_2^2 = \tfrac{1}{4}\langle J_3\rangle^2 \tag{6}$$

or, what are all the states $|w\rangle$ which verify Eq. (6)?

For a careful analysis of Eq. (6), let us define two homogeneous functionals of zeroth order, the uncertainty functional $I(\psi)$,

$$I(\psi) \equiv \langle\psi|\Delta J_1^2|\psi\rangle\langle\psi|\Delta J_2^2|\psi\rangle\langle\psi|\psi\rangle^{-2}, \tag{7a}$$

and the half-commutator squared functional $C(\psi)$,

$$C(\psi) \equiv 4^{-1}|\langle\psi|[J_1, J_2]|\psi\rangle|^2\langle\psi|\psi\rangle^{-2}. \tag{7b}$$

In terms of these functionals the Heisenberg equality looks like

$$I(\psi)=C(\psi). \tag{6'}$$

We shall refer to $|u\rangle$ as a minimum (maximum, stationary) uncertainty state if $I(\psi)$ has a local minimum (maximum, stationary point) at $|\psi\rangle=|u\rangle$. Moreover, $|w\rangle$ shall be called an intelligent state if $I(w)=C(w)$.

Therefore, in principle we have three different kind of states related to the angular momentum algebra: the Bloch states $|\tau\rangle$, the intelligent states $|w\rangle$, and the minimum uncertainty states $|u\rangle$.

It is worthwhile to stress that, in the case of the Heisenberg algebra $\{x,p,[x,p]=i\}$, the corresponding functional $C(\psi)=4^{-1}\langle\psi|[x,p]|\psi\rangle^2\cdot\langle\psi|\psi\rangle^{-2}$ has a constant value: $\frac{1}{4}$. Therefore, any intelligent state of this algebra must be a minimum uncertainty state too.

However, this property does not necessarily hold for other algebras where $C(\psi)$ is not a constant number, as in the case of SU(2).

It is a well established property of quantum mechanics[7] that all the intelligent spin states are given by the set of all the states that satisfy the linear equation,

$$J_\alpha|w\rangle\equiv(J_1-i\alpha J_2)|w\rangle=(\langle J_1\rangle_w-i\alpha\langle J_2\rangle_w)|w\rangle\equiv w|w\rangle, \tag{8a}$$

where α is a real number. Defining $\gamma_\epsilon\equiv\frac{1}{2}(1-\epsilon\alpha)$, $\epsilon=\pm1$, J_α can also be written as a linear combination of the ladder operators,

$$J_\alpha=\gamma_+J_++\gamma_-J_-=\gamma_\epsilon J_\epsilon, \tag{8b}$$

leading to the explicit expression of the intelligent spin states shown in Ref. 1. With the present notation they can be written as

$$|w_N(\tau_\alpha)\rangle=\hat{a}_N\sum_{l=0}^{N}\binom{N}{l}(2j-l)!(-2\tau_\alpha J_+)^l|\tau_\alpha\rangle,\quad 0\leqslant N\leqslant 2j,$$
$$\tau_\alpha^2=\gamma_+\gamma_-^{-1},\quad w_N\equiv 2\gamma_+\tau_\alpha^{-1}(j-N), \tag{9}$$

where $\hat{a}_N$ is a normalizing factor which shall be determined later on.

We note that for a given τ_α we have $2j+1$ different eigenvalues w_N, as we see from the explicit form of w_N. Therefore, the set $\{|w_N(\tau_\alpha)\rangle\}$ is for a given α, $|\alpha|\neq1$, a basis of H_j.[8]

It is also worthwhile to point out that, due to the fact that α must be real (therefore $\gamma_+\gamma_-^{-1}$ is real too), $\tau_\alpha=\pm(\gamma_+/\gamma_-)^{1/2}$ can only be real or pure imaginary.[9]

However, we could think of enlarging the definition (9) for $|w_N(\tau)\rangle$ to any complex number without giving raise to any mathematical inconsistency. In this case one has to stress that for complex τ not on the real or imaginary axis, $|w_N(\tau)\rangle$ does not represent a solution of the Heisenberg equation anymore. We shall call these states the generalized intelligent states.

There are two special cases of N, the extremes 0 and $2j$. In fact $|w_0(\tau)\rangle=|\tau\rangle$ and (it shall be shown in this section) $|w_{2j}(\tau)\rangle=|-\tau\rangle$. Actually these are the simpler cases of the general law relating intelligent states corresponding to opposite complex numbers,

$$|w_{N_1}(\tau_1)\rangle=|w_{N_2}(\tau_2)\rangle,\quad N_1+N_2=2j,\quad \tau_1+\tau_2=0. \tag{10}$$

This relation is easily seen after having established the value of the inner product $\langle\rho|w_{N_1}(\tau)\rangle=\langle w_0(\rho)|w_{N_1}(\tau)\rangle$ given in Appendix A.[10]

In order to perform calculations of physical interest, it is convenient to have a simpler expression than Eq. (9) to describe the intelligent states. Fortunately this can be done just by ordinary straightforward algebra. It turns out that $|w_N(\tau)\rangle$ can be written as

$$|w_n(\tau)\rangle=a_nY_1\partial_y^n y^{2j}\exp(\tau_yJ_+)|-j\rangle,$$
$$n=0,\ldots,2j, \tag{11}$$

where

$$a_n\equiv\hat{a}_NN!(1+|\tau|^2)^{-j},\quad n,\ Y_1,\ \tau_y \text{ given by}$$
$$n\equiv2j-N,\quad Y_1f(y)\equiv f(1),$$
$$(\partial_y^n)f(y)\equiv\partial^nf/\partial_y^n,\quad \tau_y\equiv\tau(1-2y^{-1}), \tag{12}$$

and the corresponding eigenvalue w_n is given by $w_n\equiv2\gamma_+\tau^{-1}(j-n)$. Taking into account definition (4) and introducing the auxiliary polynomials $p_j(y,z,|\tau|)$,

$$p_j(y,z,\tau)\equiv(yz+\tau\tau^*(y-2)(z-2))^j, \tag{13}$$

one can write down the intelligent states as

$$|w_n(\tau)\rangle=a_nY_1\partial_y^n\exp(\tau_yJ_+)\exp(-2\ln yJ_3)|-j\rangle$$
$$=a_nY_1\partial_y^np_j(y,y,\tau)|\tau_y\rangle, \tag{14a}$$

where the normalizing factor a_n is shown to be (see Appendix A)

$$a_n=\{Z_1Y_1\partial_y^n\partial_z^np_{2j}(y,z,\tau)\}^{-1/2}\equiv(p_{2j}^{nn})^{-1/2} \tag{14b}$$

and $|\tau_y\rangle$ means the Bloch state corresponding to the complex number $\tau(1-2y^{-1})=\tau_y$.

We note that in the expression given in Eq. (14a) for the intelligent spin states the operator $Y_1\partial_y^n$ occurs. Therefore, one has to know the behavior of $p_j(y,y,\tau)|\tau_y\rangle$ in a neighborhood of $y=1$, in order to obtain the corresponding derivatives.

States having the structure $p_j(y,y,\tau)|\tau_y\rangle=\exp(\tau_yJ_+)\times\exp(-2\ln yJ_3)|-j\rangle$ are not atomic coherent, since the group parameters τ_y, y do not verify the condition for a Bloch-type rotation $R(\tau)$ $(y\neq1+|\tau_y|^2)$.

However, the structure (14a) proves to be very useful in order to deduce many properties of intelligent states from the corresponding properties of the associated Bloch states $|\tau_y\rangle$.

One can also ask if an intelligent state $|w_n(\tau)\rangle$ coincides with some Bloch state $|\mu\rangle$. In order to answer this question, one can prove that[11]

$$|w_n(\tau)\rangle=|\mu\rangle\rightleftarrows n=0,\quad \tau=-\mu\quad\text{or}\quad n=2j,\quad\tau=\mu, \tag{15}$$

which shows that proper intelligent states ($|w_n(\tau)\rangle$, $n\neq0,2j$) are not Bloch states, but a refinement of them.

Moreover, since for each τ we have $2j+1$ different intelligent states, it is natural to enquire whether they could be obtained through some operation applied to the

Wigner basis $|m\rangle$. In other words: Are the $2j+1$ states $|\tau, m\rangle \equiv R(\tau)|m\rangle$ $(m=-j+1,\ldots,j-1,j)$ intelligent?

Straightforward calculation yields (notice that $\tau=\tan\frac{1}{2}\theta\exp(-il\pi/2)$, l integer)

$$J_\alpha|\tau,m\rangle = -m\sin\theta|\tau,m\rangle + \cos\theta(j+m)^{1/2}\times(j-m+1)^{1/2}|\tau,m-1\rangle. \tag{16}$$

The second term in the right-hand side shows that $|\tau,m\rangle$ is not an eigenvector of J_α, unless $\cos\theta(j+m)=0$.

As in general $|\tau|\neq 1$, the only possibility we are left with is $m=-j$, which means that in the set $\{|\tau m\rangle\}$, only $|\tau,-j\rangle=|\tau\rangle$ is intelligent. In the particular case where $\cos\theta=0$ $(\theta=\pi/2+k\pi$, k integer), it is immediate to see that such a situation corresponds to $\alpha=0$, ∞, i.e., $J_\alpha=J_1$ or J_2, respectively. In that case it is easy to understand why $|\tau=\exp(-il\pi/2),m\rangle$ is an eigenstate of J_1 (or J_2): $R(\tau=\exp(-il\pi/2)$ corresponds to $\pi/2$ rotations about J_2 (or J_1), therefore the states $|\tau\rangle=\exp(-il\pi/2),m\rangle$ are nothing else but the Wigner basis with respect to the x (or y) axis.

3. EXPECTATION VALUES FOR INTELLIGENT STATES AND MINIMUM UNCERTAINTY STATES

In order to define calculations of physical quantities for systems prepared in an intelligent state, one has to develop a suitable technique to handle the corresponding matrix elements. As ACGT have shown for the Bloch states, the technique of the generating functions has been proved to be very useful. In Appendix A we present with some details how the technique due to ACGT is extended to deal with intelligent spin states.

If we define the operators $(\cdot)^{n_1n_2}$ as

$$f^{n_1n_2}\equiv Y_1Z_1\partial_y^{n_1}\partial_z^{n_2}f(y,z)\equiv\left[\frac{\partial^{n_1}}{\partial_y^{n_1}}\frac{\partial^{n_2}}{\partial_z^{n_2}}f(y,z)\right]_{y=z=1}, \tag{17}$$

one finds (see Appendix A) for the expectation values of J_i for a system in an intelligent state,

$$\begin{aligned}
&\langle w_n(\tau)|J_1|w_n(\tau)\rangle\\
&\quad\equiv\langle J_1\rangle_{n\tau}=2j\,\mathrm{Re}\tau[y(z-2)p_{2j-1}(y,z,\tau)]^{nn}(p_{2j}^{nn})^{-1}\\
&\langle w_n(\tau)|J_2|w_n(\tau)\rangle\\
&\quad\equiv\langle J_2\rangle_{n\tau}=-2j\,\mathrm{Im}\tau[y(z-2)p_{2j-1}(y,z,\tau)]^{nn}(p_{2j}^{nn})^{-1}\\
&\langle w_n(\tau)|J_3|w_n(\tau)\rangle\\
&\quad\equiv\langle J_3\rangle_{n\tau}=j[(\tau\tau^*(y-2)(z-2)-zy)p_{2j-1}]^{nn}(p_{2j}^{nn})^{-1}
\end{aligned} \tag{18}$$

Further on, by taking second-order derivatives of the generating function X_A, defined in Eq. (A8), we evaluate the quadratic deviations $\langle\Delta J_i^2\rangle_{n\tau}$,

$$\begin{aligned}
\langle\Delta J_1^2\rangle_{n\tau}=&\tfrac{1}{2}j(2j-1)(\tau^2+\tau^{*2})[y^2(z-2)^2p_{2j-2}]^{nn}(p_{2j}^{nn})^{-1}\\
&+j[(y^2z^2+2j\tau\tau^*yz(y-2)(z-2))p_{2j-2}]^{nn}(p_{2j}^{nn})^{-1}\\
&+\tfrac{1}{2}j[(\tau\tau^*(y-2)(z-2)-zy)p_{2j-1}]^{nn}\\
&-4j^2(\mathrm{Re}\tau)^2\{[y(z-2)p_{2j-1}]^{nn}\}^2(p_{2j}^{nn})^{-2},
\end{aligned} \tag{19}$$

$$\begin{aligned}
\langle\Delta J_2^2\rangle_{n\tau}=&-\tfrac{1}{2}j(2j-1)(\tau^2+\tau^{*2})[y^2(z-2)^2p_{2j-2}]^{nn}(p_{2j}^{nn})^{-1}\\
&+j[(y^2z^2+2j\tau\tau^*yz(y-2)(z-2))p_{2j-2}]^{nn}(p_{2j}^{nn})^{-1}\\
&+\tfrac{1}{2}j[((y-2)(z-2)\tau\tau^*-zy)P_{2j-1}]^{nn}(p_{2j}^{nn})^{-1}\\
&-4j^2(\mathrm{Im}\tau)^2\{[y(z-2)P_{2j-1}]^{nn}\}^2(p_{2j}^{nn})^{-2},
\end{aligned}$$

$$\begin{aligned}
\langle\Delta J_3^2\rangle_{n\tau}=&-4j^2\{[zyp_{2j-1}]^{nn}\}^2(p_{2j}^{nn})^{-2}+2j[zyp_{2j-1}]^{nn}\\
&\times(p_{2j}^{nn})^{-1}+2j(2j-1)[y^2z^2p_{2j-2}]^{nn}(p_{2j}^{nn})^{-1}.
\end{aligned}$$

In a similar way, the mean values of monomials of the type $J_{i_1}^{a_1}J_{i_2}^{a_2}J_{i_3}^{a_3}$ can also be calculated by an appropriate number of derivatives of the generating functions, one of which is $X_A(\alpha\beta\gamma)$, defined in Eq. (A8).

Once we have obtained the values of $\langle\Delta J_{1,2}^2\rangle_{n\tau}$ and $\langle J_3\rangle_{n\tau}$ we are in a position to discuss more precisely what are the differences between minimum uncertainty states $|\mu\rangle$, and intelligent states $|w\rangle$ of the SU(2) algebra. As we know this algebra has commutators which are not numbers, it is a good candidate to find out explicit examples of intelligent states which are not minimum uncertainty states.

Actually, in order to determine all the minimum uncertainty states, one should have to parametrize H_j and thereafter calculate $I(\psi)$ and $C(\psi)$ for this H_j parametrization. Proceeding in that way, one obtains two functions depending upon $4j+1$ independent real parameters and it is a standard task to find both the local minimums of $I(\psi)$ and the subvariety where $I(\psi)=C(\psi)$.

If we restrict ourselves to a subset B of H_j, we can explore what happens on B. Evidently, any intelligent state that belongs to B is an intelligent state in H_j. On the contrary, that $|u_B\rangle$ is a minimum uncertainty state on B does not necessarily imply that $|u_B\rangle$ shall be a minimum uncertainty state on the large variety H_j.

For $B\equiv\{|\tau\rangle,\ \tau=\tan\frac{1}{2}\theta\exp(-i\varphi)\}$, the uncertainty functional $I(\psi)$ has, on B, the value[12]

$$I(\tau)=\tfrac{1}{4}j^2(1-\sin^2\theta\sin^2\varphi)(1-\sin^2\theta\cos^2\varphi), \tag{20}$$

while for $C(\psi)$, we have

$$C(\tau)=4^{-1}j^2\cos^2\theta. \tag{21}$$

Due to the simplicity of both $I(\tau)$ and $C(\tau)$, it is immediate to solve the Heisenberg equation $I(\tau)=C(\tau)$. That gives

$$j^2\sin^4\theta\sin^2 2\varphi=0, \tag{22}$$

or equivalently

$$\theta=0,\varphi\text{ arbitrary},\quad\theta\text{ arbitrary},\quad\varphi=n\pi/2\ (n\text{ integer}).$$

Because of the degeneracy at the origin in the polar representation (θ,φ) of the complex plane, the solution given in Eq. (22) is exactly the set of the two axes of the complex plane. That corresponds to the fact already mentioned: The only intelligent Bloch states are those contained in the two axes. Of course, as we have shown before, there are intelligent states which are not Bloch states.

In connection with the possible minimum uncertainty states located on B, one has to find the local minimums of $I(\tau)$. $I(\tau)$ has nine stationary points τ_s,

$$\tau_s=\tan(m\pi/4)\exp(-in\pi/4),\quad m=0,1,\quad n=0,1,\ldots,6,7. \tag{23}$$

It is straightforward to verify that $\tau_s=0$ gives a maximum of $I(\tau)$, and that $\tau_s=\exp(-in\pi/2)$ give the four minimums while the remaining four points $\tau_s=\exp[-i(\pi/4$

$+n\pi/2)]$ give saddle points of $I(\tau)$ in the subset B. That means that only the four points of $B(\tau_s = \exp(-in\pi/2))$ can be minimum uncertainty states on H_j.

Nevertheless we have a lot of intelligent states defined on B (τ any real or pure imaginary number) which shall proceed to be intelligent states when we enlarge the calculations to the whole H_j.

4. DYNAMICAL PROPERTIES OF THE INTELLIGENT STATES

The first situation that we want to consider is the time evolution of a nonrelativistic spin j system (of magnetic moment γ), in a magnetic environment $\mathbf{B}(t)$ of the type considered by Gilmore[13]:

$$\mathbf{B}(t) \equiv 2B_\perp(\cos 2\omega_1 t\hat{x} + \sin 2\omega_1 t\hat{y}) + 2B_\parallel \hat{z}, \tag{24}$$

where $2B_\parallel \hat{z}$ is a constant magnetic field along a fixed direction and B is the strength of a perpendicular field of proper frequency $2\omega_1$.

The corresponding time-dependent Hamiltonian is

$$\begin{aligned} H(t) = -\hbar\gamma \mathbf{J}\cdot\mathbf{B}(t) = -\hbar\gamma(&B_\perp \exp(-2i\omega_1 t)J_+ \\ &+ B_\perp \exp(2i\omega_1 t)J_- + 2B_\parallel J_3), \end{aligned} \tag{25}$$

with $\mathbf{J}$ represented in the $(2j+1)$-dimensional space H_j. By going to the two-dimensional representation of SU(2), Gilmore has evaluated the time evolution operator $U(t)$ which satisfies the Schrödinger equation $i\hbar\dot{U} = HU$,

$$U(t) = \begin{Bmatrix} \cos^2\psi \exp(i\omega_- t) + \sin^2\psi \exp(-i\omega_+ t) & i\sin 2\psi \sin\omega_2 t \exp(-i\omega_1 t) \\ i\sin 2\psi \sin\omega_2 t \exp(i\omega_1 t) & \cos^2\psi \exp(-i\omega_- t) + \sin^2\psi\exp(i\omega_+ t) \end{Bmatrix}, \tag{26a}$$

where ω_+, ω_-, and ψ are given by

$$\begin{aligned} &\omega_\pm \equiv \omega_2 \pm \omega_1, \quad \omega_2 \equiv [\gamma^2 B_\perp^2 + (\gamma B_\parallel + \omega_1)^2]^{1/2}, \\ &\sin 2\psi \equiv \gamma B_\perp \omega_2^{-1}, \quad \cos 2\psi \equiv (\gamma B_\parallel + \omega_1)\omega_2^{-1}. \end{aligned} \tag{26}$$

Let us assume that our system has been initially prepared in an intelligent state $|w_n(\tau)\rangle$. Therefore, in any other subsequent instant t, the system shall be in a certain state $|w_n(t,\tau)\rangle$ determined by the evaluation operator $U(t)$; namely, $|w_n(t,\tau)\rangle = U(t)|w_n(\tau)\rangle$. We want to investigate whether $|w_n(t,\tau)\rangle$ is an intelligent state or, at least, how close to an intelligent state it is while it evolves. We know, after ACGT, that a Bloch state remains a Bloch state along its evolutions under the Hamiltonian (25).

Moreover, as both $|w_0(\tau)\rangle$ and $|w_{2j}(\tau)\rangle$ are Bloch states, it might happen that any proper intelligent state could evolve remaining in the subset of the intelligent states too.

In order to give an answer to this question, let us briefly mention some useful facts concerning SU(2) and $|w_n(\tau)\rangle$, as has been given in Eq. (14a).

The first property we want to point out concerns the structure of $|w_n(\tau)\rangle$ itself; $|w_n(\tau)\rangle$ can be written

$$\begin{aligned} &|w_n(\tau)\rangle = a_n Y_1 \partial_y^n k(y,\tau)\,|-j\rangle, \\ &k(y,\tau) \equiv \exp(\tau_y J_+)\exp(-J_3 \ln y^2), \end{aligned} \tag{27}$$

where $k(y,\tau)$ belongs to $SL(2,C)$,[4] the analytic continuation of SU(2).[14] In the two-dimensional representation of $SL(2,C)$, $k(y,\tau)$ has the form

$$\begin{aligned} k(y,\tau) &= \exp(\tau_y J_+)\exp(-(\ln y^2)J_3) \\ &= \begin{Bmatrix} 1 & \tau_y \\ \cdot & 1 \end{Bmatrix} \begin{Bmatrix} y^{-1} & \cdot \\ \cdot & y \end{Bmatrix} = \begin{Bmatrix} y^{-1} & \tau(y-2) \\ \cdot & y \end{Bmatrix}, \end{aligned} \tag{28}$$

showing that it belongs to the well-known four parameter subgroup K of $SL(2,C)$,[15] as reviewed in Appendix B. We prove in this appendix that for $y \neq 1$, $k(y,\tau)$ contains a Lorentz boost and, therefore, $k(y,\tau)$ does not represent a proper rotation.

The operator $U(t)k(y,\tau) \equiv \hat{l}(t,y,\tau)$ has also been explicitly evaluated in Appendix B, Eq. (B8). This allows us to write the state $|w_n(t,\tau)\rangle$ as follows:

$$|w_n(t,\tau)\rangle = a_n(\tau) Y_1 \partial_y^n \hat{l}_4^{2j} \exp(\hat{l}_2 \hat{l}_4^{-1} J_+)\,|-j\rangle, \tag{29a}$$

where

$$\begin{aligned} \hat{l}_2 \equiv\; &[\tau(y-2)\cos^2\psi + y\sin\psi\cos\psi]\exp(i\omega_- t) \\ &+ [\tau(y-2)\sin^2\psi - y\sin\psi\cos\psi]\exp(-i\omega_+ t), \\ \hat{l}_4 \equiv\; &[\tau(y-2)\sin\psi\cos\psi + y\sin^2\psi]\exp(i\omega_+ t) \\ &+ [y\cos^2\psi - \tau(y-2)\sin\psi\cos\psi]\exp(-i\omega_- t). \end{aligned}$$

Although the structure of the state $|w_n(t,\tau)\rangle$ seems complicated, it is proved in Appendix B that this state becomes, up to a phase factor, an intelligent state if the transverse magnetic field vanishes, i.e., $B_\perp = 0$. Only in this case the evolution of an intelligent state of order n determined by the complex number τ is a generalized intelligent state, of the same order n, corresponding to the complex t-dependent number $\tau' = \tau\exp(2i\gamma B_\parallel t)$. If $n=0$ we recover the result of ACGT: $|w_0(t,\tau)\rangle = \exp(2i \times \arg\hat{l}_4)\,|\hat{l}_2 \hat{l}_4^{-1}\rangle$. That is, the evolution of a Bloch state keeps being a Bloch state, up to a phase factor.

The second situation we want to treat here is the relevance of the intelligent states in connection with the pointlike laser,[16] either with a semiclassical or a fully quantized representation of the laser field.

By a semiclassical pointlike laser we mean a collection of identical atoms, each with two effective energy levels (with $\hbar\omega$ the energy gap) interacting with a classical field $\mathbf{E}(t) = 2\mathrm{Re}\{\mathbf{E}_0 \exp(i\omega t)\}$, which has the resonant mode of frequency ω.

The Hamiltonian corresponding to this system is, following ACGT,

$$\begin{aligned} H = H_A + H_{AF} \equiv \hbar\omega J_3 &- (\mathbf{p}\cdot\mathbf{E}_0^*)J_+ \exp(-i\omega t) \\ &- (\mathbf{p}^*\cdot\mathbf{E}_0)J_-\exp(i\omega t), \end{aligned} \tag{30}$$

where the vector $\mathbf{p}$ is the complex dipole moment associated to each atom giving rise to the total dipole moment

$$\mathbf{D} \equiv \mathbf{p} J_+ + \mathbf{p}^* J_- . \tag{31}$$

For a system of N_0 atoms, the cooperation number j must satisfy the inequality

$$j \leq \tfrac{1}{2} N_0 . \tag{32}$$

We are assuming either that $\mathbf{p}$ verifies $\mathbf{p}\cdot\mathbf{E}_0 = 0 = \mathbf{p}^*\cdot\mathbf{E}_0^*$ or, if this selection rule does not apply, that we are working in the rotating wave approximation.

If one neglects the interaction term H_{AF} between matter and the electromagnetic field, namely $H_{AF}=0$ in Eq. (30), it is possible to give an estimate of the expectation value of $\mathbf{D}$. For the system initially in an intelligent state $|w_n(\tau)\rangle$, the state $|w_n(t,\tau)\rangle$ becomes $|w_n(t,\tau)\rangle = \exp(-ij\omega t)\,|w_n(\tau\exp(-ij\omega t))\rangle$. Therefore,

$$\begin{aligned}\langle w_n(t,\tau)|\mathbf{D}|w_n(t,\tau)\rangle &= \mathbf{p}\langle J_+\rangle_{n\tau}(t) + \mathbf{p}^*\langle J_-\rangle_{n\tau}(t)\\ &= 2j(\mathbf{p}\tau^*\exp(i\omega t) + \mathbf{p}^*\tau\exp(-i\omega t))\\ &\quad\times[y(z-2)p_{2j-1}]^{nn}(p_{2j}^{nn})^{-1}. \end{aligned} \tag{33}$$

This result is a refinement of the corresponding one for the Bloch state, which is reobtained here by taking $n=0$. (It is worthwhile to remind the reader that for the Wigner–Dicke states the expectation value of $\mathbf{D}$ vanishes.)

As the macroscopic dipole of the system does not vanish, there exists a nonvanishing classical radiation intensity I_c generated by this oscillating dipole, which in the wave zone is

$$I_c = I_0\cdot 4j^2\tau\tau^*\{[y(z-2)p_{2j-1}]^{nn}\}^2(p_{2j}^{nn})^{-2}. \tag{34}$$

Introducing the fully quantized Hamiltonian

$$H = H_A + H_F + H_{AF} \equiv \hbar\omega J_3 + \hbar\omega a^+ a + \gamma a J_+ + \gamma a^+ J_- , \tag{35}$$

we can calculate the emission rate for the pointlike laser.[17]

The spontaneous emission intensity can be calculated for an initial intelligent state $|w_n(\tau)\rangle$, in a way similar to what ACGT did for this model,

$$\begin{aligned} I_n^{sp} &= I_0\sum_{m=-j}^{j}|\langle m|J_-|w_n(\tau)\rangle|^2\\ &= I_0\langle w_n(\tau)|J_+J_-|w_n(\tau)\rangle\\ &= I_0\langle J_-J_+\rangle_{n\tau} + 2I_0\langle J_3\rangle_{n\tau}. \end{aligned} \tag{36a}$$

The matrix elements occurring in this relation are easily evaluated by means of the generating function $X_A(\alpha,\beta,\gamma)$, given by Eq. (A9),

$$\begin{aligned} I_{n\tau}^{sp} &= I_0 2j\tau\tau^*\{(2j-1)[yz(y-2)(z-2)p_{2j-2}]^{nn}\\ &\quad\times(p_{2j}^{nn})^{-1} + [(y-2)(z-2)p_{2j-1}]^{nn}(p_{2j}^{nn})^{-1}\}, \end{aligned} \tag{36b}$$

an expression which reduces for $n=0$ to the results found by ACGT for Bloch states.

In the case of a Dicke–Wigner initial state $|m\rangle$, the spontaneous emission intensity is $I_D^{sp} = I_0(j+m)(j-m+1)$. In order to compare the spontaneous emission intensities between intelligent states and Dicke–Wigner states we have to evaluate I_D^{sp} for a Dicke–Wigner state having the same energy expectation value that $|w_n(\tau)\rangle$.[18] Therefore, introducing $m=\langle J_3\rangle_{n\tau}$ in I_D^{sp}, we get

$$\begin{aligned} [I_D^{sp}]_{m=\langle J_3\rangle_{n\tau}} &= I_0\cdot 2j[1-(yzp_{2j-1})^{nn}(p_{2j}^{nn})^{-1}]\\ &\quad\times[1+2j(yzp_{2j-1})^{nn}(p_{2j}^{nn})^{-1}] \neq I_{n\tau}^{sp}. \end{aligned} \tag{37}$$

A similar calculation for the stimulated intensity I^{st} leads to:

$$\begin{aligned} I_{n\tau}^{st} &= I_0\sum_m\{|\langle m|J_-|w_n(\tau)\rangle|^2 - |\langle m|J_+|w_n(\tau)\rangle|^2\}\\ &= 2\langle J_3\rangle_{n\tau}\cdot I_0 . \end{aligned} \tag{38}$$

Consequently, using the value given in Eq. (18) of $\langle J_3\rangle_{n\tau}$ we have

$$I_{n\tau}^{st} = I_0\cdot 2j[1-2(yzp_{2j-1})^{nn}(p_{2j}^{nn})^{-1}] = I_D^{st}, \tag{39}$$

which is identical to the stimulated intensity emitted for an initial Dicke state with quantum number $m=\langle J_3\rangle_{n\tau}$.

Just for completeness, one can explicitly calculate $\langle J_3\rangle_{0\tau}$, $\langle J_3\rangle_{1\tau}$, and $\langle J_3\rangle_{2\tau}$. It happens that, for $j\geq 3$, the three values decrease for $0\leq\theta<\pi/2$, and increase for

$$\pi/2<\theta<\pi,$$

$$\langle J_3\rangle_{2\tau} < \langle J_3\rangle_{1\tau} < \langle J_3\rangle_{0\tau} \quad (j\geq 3), \tag{40a}$$

$$\langle J_3\rangle_{0\tau} = -2j\cos\theta, \quad \langle J_3\rangle_{1\tau} = -2j\cos\theta\left[1+\frac{(2-j^{-1})\sin^2\theta}{2j\cos^2\theta+\sin^2\theta}\right], \tag{40b}$$

$$\langle J_3\rangle_{2\tau} = -2\cos\theta\,\frac{[(j-1)(j-2)(2j-3)\cos^4\theta + 2(j-1)(4j-5)\cos^2\theta + (5j-4)]}{[(j-1)(2j-3)\cos^4\theta + 4(j-1)\cos^2\theta + 1]} \tag{40c}$$

However, as we have not been able to proceed a step further we are not allowed to claim a general property from Eqs. (40). The only statement we are making is that the stimulated emission intensity (and also the energy expectation value) of the proper intelligent states ($n=1,2$) is greater than the stimulated emission intensity arising from the Bloch state corresponding to the same value of the parameter τ.

The last point we want to mention concerning the different behavior of intelligent states in comparison with Bloch states is the following: Suppose we have initially prepared a system of spin j in an intelligent state $|w_n(\tau)\rangle$ and we want to know what is the probability that, under the magnetic Hamiltonian (25), the system could be found in $t>0$ in a Wigner state $|m\rangle$. Making use of the results of Appendices A and B, we obtain the transition probabilities

$$p_{(n,\tau)\to|m\rangle} = |\langle m|w_n(t\tau)\rangle|^2 = (n!)^2 a_n^2(\tau)\binom{2j}{j+m}$$
$$\times\left\{\sum_{(l_1,l_2)=(0,0)}^{(n,n)} a_2^{l_1} a_2^{*l_2} a_4^{n-l_1} a_4^{*n-l_2}\right.$$
$$\times c_2^{j+m-l_1} c_2^{*j+m-n+l_2} c_4^{j-m-n+l_1} c_4^{*j-m-n+l_2}$$
$$\times\binom{j+m}{l_1}\binom{j+m}{l_2}\binom{j-m}{n-l_1}\binom{j-m}{n-l_2}, \qquad (41a)$$

where a_i and c_i, $i=2,4$ are

$$a_2 \equiv \exp(-i\omega_1 t)[\tau\cos\omega_2 t + i\sin\omega_2 t(\tau\cos 2\psi + \sin 2\psi)],$$
$$a_4 \equiv \exp(i\omega_1 t)[\cos\omega_2 t + i\sin\omega_2 t(\tau\sin 2\psi - \cos 2\psi)],$$
$$c_2 \equiv \exp(-i\omega_1 t)[(\tau-2)\cos\omega_2 t + i\sin\omega_2 t((\tau-2)\cos 2\psi + \sin 2\psi)], \qquad (41b)$$
$$c_4 \equiv \exp(i\omega_1 t)[\cos\omega_2 t - i\sin\omega_2 t(\tau\sin 2\psi + \cos 2\psi)].$$

In order to see how a pure intelligent state behaves, one can take a particular case of Eqs. (41). For instance, let us choose $|m\rangle = |-j\rangle$. Making use of the above result, it turns out that $(\tau = \tan\frac{1}{2}\theta\exp(+in\pi/2))$

$$\Gamma \equiv \frac{p_{(1,\tau)\to|-j\rangle}}{p_{(0,\tau)\to|-j\rangle}} = (2j\cos^2\theta + \sin^2\theta)^{-1}$$
$$\times\frac{1+\sin^2\omega_2 t\cdot[\sin^2 2\psi(\tau^2-1) - \tau\sin 4\psi]}{1+\sin^2\omega_2 t\cdot[\sin^2 2\psi(\tau^2-1) + \tau\sin 4\psi]}. \qquad (42)$$

This ratio Γ is finite for any ψ, τ, and t unless τ takes the value $\tau'_\psi = -\cotan 2\psi$. In that case, Eq. (42) becomes

$$\Gamma \equiv \frac{p_{(1,\tau'_\psi)\to|-j\rangle}}{p_{(0,\tau'_\psi)\to|-j\rangle}} = [1 + (2j-1)\cos^2 4\psi]^{-1}$$
$$\times(1 + \sin 2\psi^{-1}\sin 6\psi\sin^2\omega_2 t)\cos^{-2}\omega_2 t, \qquad (43)$$

showing that, for $t_n = (n+\frac{1}{2})\pi/\omega_2$ the value of Γ is infinite. Consequently we see that the behavior of the proper intelligent state $|w_1(t,\tau'_\psi)\rangle$ is qualitatively different from the behavior of the Bloch state $|w_0(t,\tau'_\psi)\rangle$.

Further, as for τ'_ψ, the function $c_4(t)$ appearing in Eq. (41) has the value

$$c_4(\tau'_\psi) = \exp(i\omega_1 t)\cos\omega_2 t. \qquad (44)$$

It is clear that for instants $t_n = (2n+1)\pi/2\omega_2$ and for numbers n, m (which have to verify $n+m \leqslant j-1$)[19] the transition probability $p_{(n,\tau'_\psi)\to|m\rangle}$ vanishes with period $T_2 = \pi/\omega_2$.

Looking at the structure of the probability $p_{(n,\tau)\to|m\rangle}$, one gets two other special values of τ,

$$\tau''_\psi = 2 - \tan 2\psi, \quad \tau'''_\psi = 2. \qquad (45)$$

These values cause the periodic vanishing of $p_{(n,\tau)\to|m\rangle}$ too, now because $c_2(t)$ vanishes with the same period as above, for each instant $t'_n = n\pi/\omega_2$ and for quantum numbers n, m such that $n+1 \leqslant j+m$.

5. DISCUSSION AND COMMENTS

We have been able to establish a clear distinction between intelligent spin states, minimum uncertainty states, and Bloch states. We have shown that the generalized intelligen states constitute a refinement of the Bloch states containing them as extreme cases.

We also pointed out in Eq. (10) the symmetry in the definition of intelligent states which allow us to restrict the analysis of $|w_n(\tau)\rangle$ to any half-plane containing the origin of the whole complex plane.

Thereafter we evaluated, through the technique of the generating functions, the expectation values of both the components of the angular momentum vector and of their mean square deviations. They turned out to be rational functions of $\tau\tau^* = \tan^2\frac{1}{2}\theta$.

Moreover, by making use of some algebraic properties of the noncompact subgroup K of $SL(2,C)$ we studied some dynamical properties of the intelligent states valid both for a reasonable time dependent model of a spin-j particle in a magnetic atmosphere and for a pointlike laser.

One important result found is that for a permanent magnetic field $B = 2B_\parallel\hat{e}_3$, proper intelligent states evolve continuously in the set of generalized intelligent states. Of course, the two extreme states $(n=0, 2j)$ which are Bloch and intelligent evolve in the assembly of the complex Bloch states.

The transition probabilities, for a system prepared in an intelligent state, of becoming in time t a Wigner—Dicke state, have been computed. It turned out that there exist three values of the real parameter τ defining an intelligent state for which $p_{(n,\tau)\to|m\rangle}$ vanishes periodically.

In the case of the pointlike laser, the spontaneous and stimulated emission intensities and the macroscopic dipole of the system have also been evaluated showing again a refinement of the results obtained using Bloch states.

We have also proved that, in general, an intelligent state is not a minimum uncertainty state and we pointed out where the noncoincidence of both kind of states stems.

It is also worthwhile to note that, contrary to what has recently been asserted by Kolodziejczyk,[20] the coherent states defined by Mikhailov[21] cannot be used to explain the relationship between coherent and intelligent states, essentially because the only Mikhailov coherent state which is intelligent is, trivially, the ground state.

Finally, let us remark that Vetri's comment[22] that Radcliffe states which do not point in the z direction and are labeled "intelligent" in Ref. 1 are actually those oriented in such a way that the $\hat{n}$ axis is along $\hat{x}$ or $\hat{y}$ is precisely what Aragone, Guerri, Salamó, and Tani meant when they said that "only those Radcliffe states located on the real line or the imaginary axis are intelligent states."

APPENDIX A

In this Appendix we are going to show the details concerning some of the calculations whose results have been used in the text.

Let us recall that the states we are dealing with have been written in the form [Eqs. (13)].

$$|w_n\rangle = a_{n_1}\partial_y^{n_1}\{p_j(y,y,|\tau|)|\tau_y\rangle\}_{y=1}, \tag{A1}$$

where

$$p_j(y,z,|\tau|) \equiv [yz + |\tau|^2(y-2)(z-2)]^j. \tag{A2}$$

Suppose we are interested in computing the value of $\langle w_{n_2}|w_{n_1}\rangle$, where $|w_{n_1}\rangle$ remains as in (A1). $|w_{n_2}\rangle$ may be written

$$|w_{n_2}\rangle = a_{n_2}\partial_z^{n_2}\{p_j(z,z,|\tau|)|\tau_z\rangle\}_{z=1}, \tag{A3}$$

where instead of y we use a different variable z, in order to avoid confusion. Making the scalar product we have (p_j is real for y, z real numbers)

$$\langle w_{n_2}|w_{n_1}\rangle = a_{n_2}^* a_{n_1}\partial_z^{n_2}\partial_y^{n_1}\{p_j(y,y,\tau) \times p_j(z,z,\tau)\langle\tau_z|\tau_y\rangle\}_{z=1} \tag{A4}$$

but, by virtue of Eq. (5),

$$\begin{aligned}\langle\tau_z|\tau_y\rangle &= (1+|\tau_y|^2)^{-j}(1+|\tau_z|^2)^{-j}(1+\tau_z^*\tau_y)^{2j}\\ &\equiv y^{2j}p_j(y,y,|\tau|)^{-1}z^{2j}p_j(z,z,|\tau|)^{-1}\\ &\quad\times[1+|\tau|^2(1-2/y)(1-2/z)]^{2j}\\ &= p_j(y,y,|\tau|)^{-1}p_j(z,z,|\tau|)^{-1}p_{2j}(y,z,|\tau|).\end{aligned} \tag{A5}$$

Introducing this value of $\langle\tau_{z_2}|\tau_y\rangle$ into Eq. (A4) we get the final value of $\langle w_{n_2}|w_{n_1}\rangle$,

$$\begin{aligned}\langle w_{n_2}|w_{n_1}\rangle &= a_{n_2}^* a_{n_1}\{\partial_z^{n_2}\partial_y^{n_1}p_{2j}(y,z,|\tau|)\}_{y=1=z}\\ &= a_{n_2}^* a_{n_1}p_{2j}^{n_1 n_2}.\end{aligned} \tag{A6}$$

If we take here $n_2 = n_1$ and impose that the result found must be 1, we get the modulus of the normalizing factor a_n, as was mentioned in Eq. (14). Once we get the value of the a_n, the scalar product (A6) is completely defined,

$$\langle w_{n_2}|w_{n_1}\rangle = \frac{p_{2j}^{n_1 n_2}}{(p_{2j}^{n_1 n_1})^{1/2}(p_{2j}^{n_2 n_2})^{1/2}}. \tag{A7}$$

In order to calculate expected values of observables contained in the SO(3) algebra, it is of crucial importance to evaluate the generator function $X_A(\alpha,\beta,\gamma)$, defined in the ACGT paper as

$$X_A(\alpha,\beta,\gamma) \equiv \langle w_n|\exp(\gamma J_-)\exp(\beta J_3)\exp(\alpha J_+)|w_n\rangle. \tag{A8}$$

Introducing the form (A3) of $|w_n\rangle$ and applying the Baker–Campbell–Haussdorff formula we have that

$$\begin{aligned}X_A(\alpha,\beta,\gamma) = |a_n|^2\partial_z^n\partial_y^n\{&zy\exp(-\beta/2)\\ &+[\tau(y-2)+\alpha y][\tau^*(z-2)+\gamma z]\exp(\beta/2\}^{2j}\end{aligned} \tag{A9}$$

which, if we define the auxiliary function q in y, z, α, β, γ, by

$$\begin{aligned}q_{2j}(\alpha,\beta,\gamma,y,z,\tau) \equiv \{&zy\exp(-\beta/2)+\exp(\beta/2)\\ &\times[\tau(y-2)+\alpha y][\tau^*(z-2)+\gamma z]\}^{2j},\end{aligned} \tag{A10}$$

can be rewritten in the shorter form

$$X_A(\alpha\beta\gamma) = |a_n|^2 q_{2j}^{nn}(\alpha,\beta,\gamma). \tag{A11}$$

Once we have evaluated X_A, it is very simple to estimate the expected values of, for instance, J_1, J_2, J_3, and $(\Delta J_1)^2$, $(\Delta J_2)^2$ for the intelligent states $|w_n\rangle$.

In fact,

$$\begin{aligned}\langle w_n|J_1|w_n\rangle &= \tfrac{1}{2}\langle w_n|J_+|w_n\rangle + \tfrac{1}{2}\langle w_n|J_-|w_n\rangle\\ &= \tfrac{1}{2}(\partial_\alpha X_A)_{\alpha=\beta=\gamma=0} + \tfrac{1}{2}(\partial_\gamma X_A)_{\alpha=\beta=\gamma=0},\end{aligned} \tag{A12}$$

$$\langle w_n|J_3|w_n\rangle = (\partial_\beta X_A)_{\alpha=\beta=\gamma=0}, \tag{A13}$$

and

$$4(\Delta J_1)^2 = \langle J_+^2\rangle + \langle J_-^2\rangle + 2\langle J_- J_+\rangle + 2\langle J_3\rangle - 4\langle J_1\rangle^2.$$

Consequently,

$$\begin{aligned}4(\Delta J_1)^2 = {}&(\partial^2_{\alpha\alpha}X_A)_{\alpha=\beta=\gamma=0} + (\partial^2_{\gamma\gamma}X_A)_{\alpha=\beta=\gamma=0}\\ &+2(\partial^2_{\alpha\gamma}X_A)_{\alpha=\beta=\gamma=0} + 2(\partial_\beta X_A)_{\alpha=\beta=\gamma=0}\\ &-[(\partial_\alpha X_A)_{\alpha=\beta=\gamma=0} + (\partial_\gamma X_A)_{\alpha=\beta=\gamma=0}]^2,\end{aligned} \tag{A14}$$

and in the same way the value of $4(\Delta J_2)^2$ can be given,

$$\begin{aligned}4(\Delta J_2)^2 = {}&-(\partial^2_{\alpha\alpha}X_A)_0 - (\partial^2_{\gamma\gamma}X_A)_0 + 2(\partial^2_{\alpha\gamma}X_A)_0\\ &+2(\partial_\beta X_\alpha)_0 + [(\partial_\alpha X_A)_0 - (\partial_\gamma X_A)_0]^2.\end{aligned} \tag{A15}$$

It is interesting to observe that $q_{2j}(0,0,0) = p_{2j}(y,z,|\tau|)$.

APPENDIX B

In this Appendix we shall give some group results concerning $SL(2,C)$ and its subgroup K.

The four-parameter subgroup K has been extensively used in connection with the irreducible representations of the Lorentz group (see for instance Ref. 15). K is defined as the set of all the elements k of $SL(2,C)$ of the form

$$K = k(p,q) = \begin{pmatrix} p^{-1} & q \\ 0 & p\end{pmatrix}, \quad p, q \text{ complex numbers.} \tag{B1}$$

The importance of K lies in the fact that any element l of $SL(2,C)$ can uniquely be decomposed in the form

$$l = kz, \quad k = \begin{pmatrix} p^{-1} & q \\ 0 & p\end{pmatrix}, \quad z = \begin{pmatrix} 1 & \cdot \\ z & 1\end{pmatrix}. \tag{B2}$$

Moreover, as any $k(pq)$ can uniquely be factorized in the form

$$\begin{aligned}k &= \begin{pmatrix} 1 & qp^{-1} \\ \cdot & 1\end{pmatrix}\begin{pmatrix} p^{-1} & \cdot \\ \cdot & p\end{pmatrix} = \begin{pmatrix} p^{-1} & q \\ \cdot & p\end{pmatrix}\\ &= \exp(qp^{-1}J_+)\exp(-2\ln p J_3),\end{aligned} \tag{B3}$$

l can be uniquely decomposed as a product of three exponentials,

$$l = \exp(qp^{-1}J_+)\exp(-2\ln p\cdot J_3)\exp(zJ_-).$$

Let an arbitrary $l \in SL(2,C)$ be given,

$$l = \begin{pmatrix} l_1 & l_2 \\ l_3 & l_4\end{pmatrix}. \tag{B4}$$

It is easy to check that

$$l = \exp(l_2 l_4^{-1}J_+)\exp(-2\ln l_4 J_3)\exp(l_3 l_4^{-1}J_-). \tag{B5}$$

The elements $k(y,\tau)$ defined in Eq. (28) have the structure (B1), therefore the convenience of dealing with K (even if the restriction one could make of keeping p real could suggest that the three-dimensional subgroup $K' \equiv \{k \in K : p \text{ real}\}$ should play some specific role, more centrally than K itself).

Just for the sake of completeness it is possible to write down the four-dimensional Lorentz transformation $\Lambda(k)$ represented by $k(y,\tau)$. Following Gel'fand, Graev, and Vilenkin[23] it is straightforward to prove that $\Lambda(k) = \Lambda_1\Lambda_2$, where Λ_1 is the standard Lorentz boost ($|y| \neq 1$) and Λ_2 is a distortion of the $\{x^2, x^-\}$ two-dimensional plane (or of the $\{x^1, x^-\}$ two-plane accordingly to whether τ is a real or an imaginary number, respectively). The distortion Λ_2 turns out to be

$$\begin{aligned}(\Lambda_2 x)^+ &= x^+,\\ (\Lambda_2 x)^- &= x^- + \tau_y\tau_y^* x^+ + 2^{1/2}\mathrm{Re}\tau_y x^2 - 2^{1/2}\mathrm{Im}\tau_y x^1,\\ (\Lambda_2 x)^1 &= x^1 - 2^{1/2}\mathrm{Im}\tau_y x^+, \quad (\Lambda_2 x)^2 = x^2 + 2^{1/2}\mathrm{Re}\tau_y x^+,\end{aligned} \tag{B6}$$

while the boost Λ_1 applied to $\hat{x} \equiv \Lambda_2 x$ gives

$$\begin{aligned}(\Lambda_1\hat{x})^+ &= y^{-2}\hat{x}^+, \quad (\Lambda_1\hat{x})^- = y^2\hat{x}^-,\\ (\Lambda_1\hat{x})_1 &= \hat{x}_1, \quad (\Lambda_1\hat{x})_2 = \hat{x}_2,\end{aligned} \tag{B7}$$

where we denoted by $x^\pm \equiv 2^{-1/2}(x^0 \mp x^3)$ the usual two null coordinates.

We are interested in the decomposition (B5) for the operator $U(t)k(y,\tau)$ in order to have $|w_n(t,\tau)\rangle$ written in a way resembling an intelligent state. Calculating the matrix product, we get

$$\hat{l} \equiv U(t)k(y,\tau) \equiv \begin{pmatrix}\hat{l}_1 & \hat{l}_2\\ \hat{l}_3 & \hat{l}_4\end{pmatrix} \equiv \begin{pmatrix} y^{-1}\cos^2\psi\exp(i\omega_- t) + y^{-1}\sin^2\psi\exp(-i\omega_+ t), & [\tau(y-2)\cos^2\psi + y\sin\psi\cos\psi]\exp(i\omega_- t) \\ & \quad + [\tau(y-2)\cos^2\psi - y\sin\psi\cos\psi]\exp(i\omega_+ t), \\ y^{-1}\sin\psi\cos\psi[\exp(i\omega_+ t) - \exp(-i\omega_- t)], & [y\sin^2\psi + \tau(y-2)\sin\psi\cos\psi]\exp(i\omega_+ t) \\ & \quad + [y\cos^2\psi - \tau(y-2)\sin\psi\cos\psi]\exp(-i\omega_- t). \end{pmatrix} \tag{B8}$$

With this result, one obtains for $|w_n(t,\tau)\rangle = U(t)|w_n(\tau)\rangle$,

$$\begin{aligned}|w_n(t,\tau)\rangle &= a_n(\tau)Y_1\partial_y^n\hat{l}(y,t,\tau)|-j\rangle\\ &= a_nY_1\partial_y^n\hat{l}_4^{2j}\exp(\hat{l}_2\hat{l}_4^{-1}J_+|-j\rangle,\end{aligned} \tag{B9}$$

or what is the same,

$$|w_n(t,\tau)\rangle = a_n(\tau)Y_1\partial_y^n\{\exp(2ij\arg\hat{l}_4) \times (|\hat{l}_2|^2 + |\hat{l}_4|^2)^j|\hat{l}_2\hat{l}_4^{-1}\rangle\}, \tag{B10}$$

in terms of the Bloch state $|\hat{\tau}\rangle = |\hat{l}_2\hat{l}_4^{-1}\rangle$. In the case where $n=0$ (and consequently the term has been prepared in a Bloch state), we have for $|w_n(t,\tau)\rangle$,

$$|w_0(t,\tau)\rangle = \exp(2ijY_1\arg\hat{l}_4)\cdot Y_1|\hat{l}_2\hat{l}_4^{-1}\rangle, \tag{B11}$$

a state which differs by a phase factor $2jY_1\arg l_4$ from the standard Bloch state corresponding to the complex number $\tau(t) = Y_1(\hat{l}_2\hat{l}_4^{-1})$.

The explicit expression shown in Eq. (B10) for $|w_n(t,\tau)\rangle$ allows an easy calculation of the transition number $\langle\mu|w_n(t,\tau)\rangle$ for an arbitrary coherent spin state $|\mu\rangle$,

$$\begin{aligned}\langle\mu|w_n(t,\tau)\rangle &= a_n(\tau)a_0(\mu)Y_1\partial_y^n\{(\hat{l}_4 + \mu^*\hat{l}_2)^{2j}\}\\ &= a_n(\tau)a_0(\mu)(!n)\binom{2j}{n}[\sin\psi(\sin\psi + \tau\cos\psi)\\ &\quad\times\exp(i\omega_+ t) + \cos\psi(\cos\psi - \tau\sin\psi)\exp(-i\omega_- t)\\ &\quad+\mu^*\sin\psi(\sin\psi\tau - \cos\psi)\exp(-i\omega_+ t)\\ &\quad+\mu^*\cos\psi(\tau\cos\psi + \sin\psi)\exp(i\omega_- t)]^n\\ &\quad\times[\sin\psi(\sin\psi - \tau\cos\psi)\exp(i\omega_+ t)\\ &\quad+\cos\psi(\cos\psi + \tau\sin\psi)\exp(-i\omega_- t)\\ &\quad-\mu^*\sin\psi(\tau\sin\psi + \cos\psi)\exp(-i\omega_+ t)\\ &\quad-\mu^*\cos\psi(\tau\cos\psi - \sin\psi)\exp(i\omega_- t)]^{2j-n'}.\end{aligned} \tag{B12}$$

This expression is very useful in order to investigate under what conditions $|w_n(t,\tau)\rangle$ could be an intelligent state. It is sufficient to calculate $\langle\mu|w_n'(\tau')\rangle$ and to compare its value with (B12). If we prove that there exists (n',τ') such that for any complex μ, $\langle\mu|w_{n'}(\tau')\rangle = \langle\mu|w_n(t,\tau)\rangle$, then the state $|w_n(t,\tau)\rangle$ keeps being intelligent along its evolution under the influence of the Hamiltonian given in Eq. (26). Since

$$\langle\mu|w_{n'}(\tau')\rangle = a_{n'}(\tau')a_0(\mu)(!n')\binom{2j}{n''}(1+\mu^*\tau')^{n'}(1-\mu^*\tau')^{2j-n'}, \tag{B13}$$

and both polynomials in the variable μ^* (B12) and (B13) must be identical, they have to contain the same roots with the same multiplicity. Therefore, n' has to be equal to n. Moreover, if we proceed with the analysis, one can immediately recognize that they are going to coincide iff $\sin 2\psi = 0$. That implies $\cos 2\psi = (-1)^p$ or, equivalently, $\psi = n\pi/2$. The condition $\psi = n\pi/2$ [see Eq. (26b)] is equivalent to saying that $B_\perp = 0$. Thus, after Eq. (B8), we have

$$|w_n(t,\tau)\rangle = \exp(-2ij\gamma B_\parallel t)|w_n(\tau\exp(2i\gamma B_\parallel t))\rangle. \tag{B14}$$

Of course, if $\tau = |\tau|\exp(in\pi/2)$, $\tau' = \tau\exp(2i\gamma B_\parallel t) = |\tau|\exp(i(n\pi/2 + 2\gamma B_\parallel t))$ we get a generalized intelligent state, which is strictly intelligent for t such that $2\gamma B_\parallel t = m\pi/2$, i.e., it is periodically intelligent.

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[8]In the sense that they are $2j+1$ nonvanishing linearly independent vectors. Of course we do not know whether they are orthogonal. Their inner product is given in Appendix A.

[9]Actually it can be easily seen that τ_α ranges over all the points different from the origin of the two axes of the complex plane while α takes any real value $\alpha : |\alpha| \neq 1$. It is easy to see that $|\alpha| = 1$ only gives a trivial solution to Eq. (8a): If $\alpha = +1$, $w = 0$, $|w_N\rangle = |-j\rangle$, and if $\alpha = -1$, $w = 0$, $|w_N\rangle = |j\rangle$. If one wants to extend the definition of $|w_N(\tau)\rangle$ to $\tau = 0$, it turns out that $|w_N(0)\rangle = |-j\rangle$ for all N.

[10]Even directly, it is enough to realize that $|w_{N_1}(\tau_1)\rangle$ and $|w_{N_2}(\tau_2)\rangle$ are normalized eigenvectors corresponding to the same eigenvalue of $J\alpha$ and that both of them have the same signature for their projection along $|-j\rangle$.

[11]In fact, take from Eq. (9), $|w_n(\tau)\rangle = a_n(2j-n)!^{-1}(1+\tau\tau^*)^j \times \sum_{l=0}^{2j-n} \binom{2j-n}{l} (2j-l)!\,(-2\tau J_+)^l\,|\tau|\rangle = |\mu\rangle = (1+\mu\mu^*)^{-j}\exp(\mu J_+)\,|-j\rangle$. Then, if we multiply both sides times $\exp(\tau J_+)$ and compare them, we see that (n, τ) has to be either $(0, -\mu)$ or $(2j, \mu)$.

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