Orthogonality:

$$\int_0^\infty dx\, \mathcal{L}_k^\alpha(x)\mathcal{L}_{k'}^\alpha(x)=\delta_{kk'}. \tag{A.15}$$

Differential equation:

$$\left(-\Lambda_\alpha^2+\frac{2k+\alpha+1}{2x}-\frac{1}{4}\right)\mathcal{L}_k^\alpha(x)=0, \tag{A.16}$$

$$\Lambda_\alpha^2\equiv-\frac{d^2}{dx^2}-\frac{1}{x}\frac{d}{dx}+\frac{\alpha^2}{4x^2}. \tag{A.17}$$

Integrals:

$$\int_0^\infty x^p e^{-x} L_k^\alpha(x) L_{k'}^{\alpha'}(x)\,dx$$

$$=\Gamma(p+1)\sum_s(-1)^{k+k'+s}\binom{p-\alpha}{k-s}\binom{p-\alpha'}{k'-s}\binom{-p-1}{s}, \tag{A.18}$$

[Re $p>-1$] (Schrödinger [9]).

$$\int_0^\infty dx\, e^{-bx}x^\alpha L_k^\alpha(\lambda x)L_{k'}^\alpha(\mu x)$$

$$=\frac{\Gamma(k+k'+\alpha+1)}{k!k'!}\frac{(b-\lambda)^k(b-\mu)^{k'}}{b^{k+k'+\alpha+1}}$$

$$\times{}_2F_1\left(-k,-k';-k-k'-\alpha;\frac{b(b-\lambda-\mu)}{(b-\lambda)(b-\mu)}\right), \tag{A.19}$$

[Re $\alpha>-1$, Re $b>0$] (Erdélyi *et al.* [16]).

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TOPIC 7. UNCERTAINTY RELATIONS FOR ANGULAR MOMENTUM[1]

1. The Role of the Uncertainty Relations

Heisenberg's [1] discovery of the uncertainty relations for position and momentum measurements in quantum mechanics played a fundamental role in clarifying the physical content of quantum mechanics (see Refs. [2] and [3]). Using the terminology devised by Dirac, one says that quantum mechanics considers two types of observables: commuting classical-type observables ("c-numbers") and quantal-type observables ("q-numbers"), the latter obeying noncommutative, but associative, algebraic rules. The basic quantal observables for a particle—position **x** and momentum **p**—are postulated in quantum mechanics to satisfy the Heisenberg[2] commutation

[1]We thank Professor Michael Reed and Dr. Michael Nieto for discussions on the content of this Topic.

[2]Alfred Landé has mentioned that in point of fact it was Born, not Heisenberg, who first wrote out these commutation rules in their final form (Heisenberg having found the diagonal elements); but the concept (if not the result) is indisputably Heisenberg's. Hence, we stick to the common usage. The historical context for the uncertainty relation is discussed in the eloquent Heisenberg memorial lecture by Mehra [5a].

rule (Heisenberg [4], Born *et al.* [5]):

$$[p_i, x_j] = -i\hbar\delta_{ij}1. \tag{5.7.1}$$

The existence of a nonvanishing commutator between two observables implies a *restriction* on the possibility of preparing quantal states in which the two observables simultaneously take definite values. The Heisenberg uncertainty relation for position and momentum measurements,

$$\Delta p_i \Delta x_i \geqslant \tfrac{1}{2}\hbar, \tag{5.7.2}$$

gives a precise statement of this restriction, which asserts an inequality relating the two dispersions, Δp_i and Δx_i, that occur for measurements on a physical system in the state $|\psi\rangle$:

$$(\Delta p_i)^2 \equiv \langle\psi|(p_i - \langle p_i\rangle)^2|\psi\rangle,$$
$$(\Delta x_i)^2 \equiv \langle\psi|(x_i - \langle x_i\rangle)^2|\psi\rangle, \tag{5.7.3}$$

where the expectation value, $\langle A\rangle$, of an observable A for a system in state $|\psi\rangle$ is defined by $\langle A\rangle = \langle\psi|A|\psi\rangle$.[1]

From the Heisenberg uncertainty relation, Eq. (5.7.2), it is clear that a position measurement of great accuracy, having small dispersion $\Delta x \sim \varepsilon$ ($\Delta x \doteq 0$ is both physically and mathematically excluded), necessarily requires a very large uncertainty $\Delta p \sim \hbar/\varepsilon$ for a simultaneous momentum measurement. It follows that in quantum mechanics—in sharp contrast to classical mechanics—there can be no meaning to such a concept as "the path of a particle." Bohr's Principle of Complementarity, formulated at the Solvay Conference[2] of 1927, was an attempt to capture the qualitative implications of the uncertainty relation. This principle asserts that atomic phenomena cannot be described with the completeness demanded by classical mechanics; some of the elements in a classical description (particle versus wave nature—that is, position versus momentum aspects) are actually mutually exclusive, but these complementary elements are all essential in the description of the phenomena. This principle is a basic tenet of the Copenhagen interpretation of quantum mechanics, which is the standard viewpoint of essentially all physicists (see Jammer [6]).

[1] The general definition of the dispersion ΔA of an observable A measured in the state $|\psi\rangle$ is $(\Delta A)^2 = \langle\psi|(A - \langle A\rangle)^2|\psi\rangle$, where $\langle A\rangle = \langle\psi|A|\psi\rangle$. Note then that, in general, $\langle A\rangle$ is a real number that depends on ψ, and ΔA is a nonnegative real number that depends on ψ. It is customary in the physics literature to suppress this (functional) dependence on the state. However, note that in Eq. (5.7.2) the right-hand side (the minimum value $\hbar/2$) *is* independent of the state $|\psi\rangle$.

[2] This conference marked the beginning of the Einstein–Bohr controversy over quantum mechanics (Robertson [3, pp. 143ff.]).

The uncertainty relations are accordingly one of the essential elements in interpreting and understanding the physical content of quantum mechanics (see Note 1). The purpose of the present Topic is to discuss in a precise mathematical way the uncertainty relations for angular momentum observables. This is clearly of intrinsic interest, but it is a task of particular importance, since the literature contains many confusions and errors. To do this properly it is useful to review first the situation for position–momentum, to which we now turn.

2. Résumé of the Position–Momentum Uncertainty Relation

The essential fact to recognize in any attempt to discuss the uncertainty relation [Eq. (5.7.2)] with precision is that the operators (observables) entering Eqs. (5.7.1) and (5.7.2) are *unbounded*: Both $p = p_i$ and $x = x_i$ have the real line as spectrum. It is a consequence of the Toeplitz–Hellinger theorem[1] that unbounded operators can be defined, at best, only on a dense subset of Hilbert space. This unfortunate fact of life is the source of much difficulty (and suffering) in quantum physics and a chief stock-in-trade of mathematical physics (see Notes 2 and 3).

There is an elegant way to avoid some, but not all, of these difficulties in the case at hand. This is to use Weyl's idea of replacing the unbounded operators p and x by the unitary (and, hence, bounded) operators defined by

$$U = \exp(i\lambda p/\hbar),$$
$$V = \exp(i\mu x/\hbar). \tag{5.7.4}$$

One thereby obtains *the Heisenberg commutation relation in the Weyl form* [8, p. 273]:

$$UV = e^{(i\lambda\mu/\hbar)}VU. \tag{5.7.5}$$

Operators in the Weyl form do not in general suffice for the purposes of physics (U and V, for example, are technically not observables). To justify the (standard) physicist's manipulations to follow, we shall assume that there exists a (dense) domain invariant under p and x, and in the domains of both.

To establish the uncertainty relation, Eq. (5.7.2), let us fix an arbitrary (normalized) Hilbert space vector $|\psi\rangle$ and consider the operators P and Q

[1] The Toeplitz–Hellinger theorem (Reed and Simon [7]) asserts that a symmetric operator [an operator A that satisfies $(A\psi, \psi) = (\psi, A\psi)$] defined on *all* vectors ψ in a Hilbert space is necessarily bounded. The term "Hermitian" is used interchangeably with "symmetric."

UV

defined by[1]

$$P \equiv p - \langle \psi | p | \psi \rangle,$$
$$Q \equiv x - \langle \psi | x | \psi \rangle. \tag{5.7.6}$$

Applying the Schwartz inequality to the vectors $P|\psi\rangle$ and $Q|\psi\rangle$, one finds

$$\langle P\psi | P\psi \rangle \langle Q\psi | Q\psi \rangle \geqslant |\langle P\psi | Q\psi \rangle|^2. \tag{5.7.7}$$

To evaluate the right-hand side in this relation, we note that

$$\begin{aligned} |\langle P\psi | Q\psi \rangle|^2 &= |\langle \psi | PQ | \psi \rangle|^2 \\ &= \tfrac{1}{4}|\langle \psi | [P, Q] | \psi \rangle + \langle \psi | PQ + QP | \psi \rangle|^2 \\ &\geqslant \tfrac{1}{4}|\langle \psi | [P, Q] | \psi \rangle|^2 \geqslant \tfrac{1}{4}\hbar^2, \end{aligned} \tag{5.7.8}$$

since $[P, Q] = -i\hbar\mathbf{1}$.

This result establishes the uncertainty relation[2] of Eq. (5.7.2), since the dispersion $(\Delta x)^2$ is given by $(\Delta x)^2 = \langle Q\psi | Q\psi \rangle$, and, similarly, $(\Delta p)^2 = \langle P\psi | P\psi \rangle$.

It is interesting to pose the question: What is the class of states for which the uncertainty relation actually achieves the minimum? These are the *minimum uncertainty states* that were defined[3] by Schrödinger [10] and that play a major role in quantum optics (Glauber [11], Klauder and Sudarshan [12], Louisell [13]). There are two conditions to be fulfilled: (*a*) For the Schwartz inequality [Eq. (5.7.7)] to be an *equality* requires that the vector $P|\psi\rangle$ be a multiple of the vector $Q|\psi\rangle$; that is,

$$P|\psi\rangle = \lambda Q|\psi\rangle, \qquad \lambda \in \mathbb{C}; \tag{5.7.9}$$

and (*b*) for Eq. (5.7.8) to be an equality requires that

$$\langle \psi | (PQ + QP) | \psi \rangle = 0. \tag{5.7.10}$$

Consider Eq. (5.7.10) first. Using $PQ = QP - i\hbar\mathbf{1}$ and then $QP = PQ + i\hbar\mathbf{1}$,

[1]Observe that these operators are functionals of ψ (see footnote 1, p. 308).

[2]This conclusion is inescapable for p and x Hermitian operators defined on a dense subset of a Hilbert space with the assumed domain properties. (The proof we have given is essentially that of von Neumann [9, pp. 230ff.]. It follows that an eigenvector of p or x cannot be a vector in Hilbert space, since such an eigenvector would satisfy $P|\psi\rangle = 0$ or $Q|\psi\rangle = 0$, thus contradicting Eq. (5.7.8). The conclusion is: The commutation relation (5.7.1) is a valid operator identity only when applied to vectors in a common dense invariant domain of Hilbert space; it is clearly invalid when acting on eigenvectors of p or x. (This is discussed further in Section 3.)

[3]Von Neumann [9, p. 237] attributes these states to Heisenberg.

we find that a minimum uncertainty state must satisfy the two conditions expressed by

$$\langle \psi | PQ | \psi \rangle = -i\hbar/2 \qquad \text{and} \qquad \langle \psi | QP | \psi \rangle = i\hbar/2. \tag{5.7.11}$$

Using next the requirement (5.7.9), we find the relations[1]

$$(\Delta p)^2 = -i\hbar\lambda/2, \qquad (\Delta x)^2 = i\hbar/2\lambda. \tag{5.7.12}$$

Note that we recover from these two relations the minimum uncertainty relation

$$\Delta p \Delta x = \hbar/2, \tag{5.7.13}$$

as well as the relation

$$\Delta p / \Delta x = -i\lambda. \tag{5.7.14}$$

Conversely, these two relations imply Eqs. (5.7.12).

Using Eq. (5.7.14) in Eq. (5.7.9), we now obtain the following equation that must be satisfied by a minimum uncertainty state $|\psi\rangle$:

$$[(\Delta p)^{-1}P - i(\Delta x)^{-1}Q]|\psi\rangle = 0. \tag{5.7.15}$$

Using the Schrödinger realization of the operators p and x [$p \to -i\hbar(\partial/\partial x)$, and x is the multiplication operator by x], we can integrate the relation (5.7.15) to obtain the following generic form for every position–momentum minimum uncertainty state, $\psi(x) = \langle x | \psi \rangle$:

$$\psi(x) = [\pi\hbar(\Delta x/\Delta p)]^{-\frac{1}{4}} \exp\left[-\left(\frac{(x - \langle x \rangle)^2}{2\hbar(\Delta x/\Delta p)} + \frac{i}{\hbar}\langle p \rangle x\right)\right]. \tag{5.7.16}$$

At first glance this result is a curious one, since $\langle p \rangle$, $\langle x \rangle$, and $\Delta x/\Delta p$ are properly to be thought of as functionals of ψ itself. On the other hand, the relationship is self-reproducing in the sense that one may set $\langle p \rangle = \hbar b (-\infty < b < \infty)$, $\langle x \rangle = a$, $(-\infty < a < \infty)$, and $\hbar(\Delta x/\Delta p) = \mu$ $(0 < \mu < \infty)$ in the right-hand side of Eq. (5.7.16), and then calculate by direct integration the values $\langle p \rangle = b\hbar$, $\langle x \rangle = a$, and $\hbar\Delta x/\Delta p = \mu$, using $\psi(x) = \psi(a, b, \mu; x)$. This result shows that, in fact, $|\psi\rangle$ *is to be interpreted as a three-parameter*

[1]Note then that we must have $\lambda = ig(\psi)$, where $0 < g(\psi) < \infty$ for all states ψ in the Hilbert space of states of a physical system (see footnote 1, p. 308).

$|A+B|^2 \leq |A|^2 + |B|^2$

family of states

$$\psi(a,b,\mu;x)=(\pi\mu)^{-\frac{1}{4}}\exp\left[-\left(\frac{(x-a)^2}{2\mu}+ibx\right)\right], \qquad (5.7.17)$$

where the real parameters a, b, and μ may assume any values such that $-\infty<a<\infty$, $-\infty<b<\infty$, and $0<\mu<\infty$.

Since the minimum uncertainty state (5.7.17) is generic—that is, it refers to no particular (one-dimensional) physical system—it is valid for every such physical system. Each member of this family is a minimum uncertainty state for any physical system and may be realized for that system by suitable (position–momentum) measurements.

The family of minimum uncertainty states (5.7.17) is in one-to-one correspondence with the ground states (lowest energy states) of a *family of harmonic oscillators*. [This statement is to be contrasted with the less carefully phrased one often found in the literature, which (misleadingly) asserts that the states (5.7.17) are "harmonic oscillator states."]

To see more clearly the relationship of the minimum uncertainty states (5.7.17) to the harmonic oscillator, we operate on Eq. (5.7.15) from the left with $(2m)^{-1}[(\Delta p)^{-1}P+i(\Delta x)^{-1}Q]$. The result is

$$\left[\frac{1}{2m}(p-\hbar b)^2+\frac{\hbar^2}{2m\mu^2}(x-a)^2\right]|\psi(a,b,\mu)\rangle=\frac{\hbar^2}{2m\mu}|\psi(a,b,\mu)\rangle. \qquad (5.7.18)$$

The operator in the left-hand side of this equation is the Hamiltonian for a particle of mass m in a harmonic oscillator potential centered at $x=a$ and observed from a reference frame moving in the x-direction with momentum $\hbar b$. Moreover, the frequency of the (classical) motion is $\omega=\hbar/m\mu$, so that the energy of the oscillator [right-hand side of Eq. (5.7.18)] is $\hbar\omega/2$. Thus, $|\psi(a,b,\mu)\rangle$ is the ground state of a harmonic oscillator with fixed physical characteristics (as determined by the mass m and the frequency $\omega=\hbar/m\mu$), which has equilibrium position $x=a$ and is observed in a moving reference frame having momentum $\hbar b$.

It is important to note that *each* member of the family of minimum uncertainty states $\{|\psi(a,b,\mu')\rangle : 0<\mu'<\infty\}$ is a minimum uncertainty state of the oscillator described above with mass m and frequency $\omega=\hbar/m\mu'$; only *one* member $(\mu'=\mu)$ of the family coincides with the ground state of the oscillator with physical parameters (m,μ). (Coherent states for a *particular* oscillator are a subset of the position–momentum minimum uncertainty states.)

We refer to the literature for further discussion of minimum uncertainty states and coherent states for other physical systems (Nieto and Simmons [14], Perelomov [15]; a recent review is by Santhanam [15a]).

3. Uncertainty Relations for Angular Momentum[1] in Two Dimensions

It is customary (in textbook treatments) to discuss the uncertainty relations for angular momentum by considering in effect only the special case of the subgroup of rotations generated by the orbital rotation operator L_3. Such a restriction limits the discussion essentially to (two-dimensional) planar rotations, a very special case indeed, which (as will be shown below) grossly distorts the actual three-dimensional situation. Since the treatment of rotations in two dimensions is, however, important in its own right, and, moreover, illustrates rather clearly certain typical *technical* difficulties, it is helpful to discuss this case first.

We begin therefore with the orbital rotation operator L_3 defined by

$$L_3=-i\left(x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}\right), \qquad (5.7.19)$$

and seek to determine the position variable that is conjugate to L_3. Let us define the angle ϕ by

$$\phi=\tan^{-1}(x_2/x_1); \qquad (5.7.20)$$

that is, we take ϕ to be the angle of rotation in the (x_1-x_2)-plane measuring the position $\mathbf{x}=x_1\hat{e}_1+x_2\hat{e}_2$ of a particle initially at the point $(r,0)$, where $r=(x_1^2+x_2^2)^{\frac{1}{2}}$.

One can verify—at this stage heuristically—that L_3 and ϕ apparently do satisfy the desired canonical commutation rule

$$[L_3,\phi]=-i1, \qquad (5.7.21)$$

since L_3 has the Schrödinger realization

$$L_3=-i(\partial/\partial\phi). \qquad (5.7.22)$$

[1]Angular momentum states which minimize the uncertainty relation $\Delta J_1\Delta J_2\geq|\langle J_3\rangle|/2$ have also been called "minimum uncertainty states" and discussed in the literature (see Bacry [15b], and references cited there). We are concerned here, however, with states minimizing the uncertainty for an operator and its conjugate observable, as in the prototype Heisenberg relation.

This conclusion agrees intuitively with one's physical understanding of planar rotations, and it agrees with the Dirac quantization prescription whereby the Poisson bracket (PB) is mapped into the quantum mechanical commutator:[1]

$$[L_3,\phi]_{PB} \to (i\hbar)^{-1}[L_3,\phi].$$

The problem one faces in a precise treatment is to *postulate* Eq. (5.7.21) as valid, and then *to determine the exact sense in which the operators L_3 and ϕ—especially the appropriate space on which they act—are to be interpreted.*

That this is not simply a matter of overly fastidious mathematical taste can be seen from the following "fallacy": Taking matrix elements of Eq. (5.7.21) between eigenstates of L_3, one obtains the result $(m'-m)\langle m'|\phi|m\rangle = \delta_{m'm}$, which is absurd, since it implies (for $m=m'$) that $0=1$. This "fallacy" has been known since the beginning of quantum mechanics (Jordan [16]) and is continually being rediscovered (for example, see Perlman and Troup [17]). The resolution of this situation is *not* to deny the commutation rule (as in Ref. [17]), but, as emphasized above, to find the precise conditions for its validity (Kraus [18]).

We begin by observing that L_3 is unbounded, and hence defined only on a dense set of Hilbert space (see Note 2). Next we observe that the most suitable function space for the present problem is the space $AC[0,2\pi)$ of absolutely continuous square-integrable functions on the interval $[0,2\pi)$ (see Note 4). For such a space the fundamental theorem of the calculus is valid, and for each function f in the space there exists a derivative f' almost everywhere. The operator L_3 is then defined as the operator acting on functions $f\in AC[0,2\pi)$ given by

$$L_3: f(\phi) \to -i\frac{df(\phi)}{d\phi}, \qquad f\in AC[0,2\pi), \qquad f(0)=f(2\pi)=0. \tag{5.7.23}$$

Here the domain of definition for the operator L_3 (denoted $\mathfrak{D}$) is the set of functions defined by $\mathfrak{D}=\{f\in AC[0,2\pi): f(0)=f(2\pi)=0\}$. The domain $\mathfrak{D}$ is dense in $AC[0,2\pi)$, and therefore Eq. (5.7.23) constitutes an adequate definition for L_3.

If we interpret the operator ϕ as the multiplication operator in $AC[0,2\pi)$ defined by

$$\phi: f(\phi) \to \phi f(\phi), \tag{5.7.24}$$

[1]Recall that L_3 in Eq. (5.7.19) is in units of $\hbar$, so that these results are dimensionally in accord.

then ϕ is a bounded operator, defined everywhere. In particular, the domain of the product of the two operators, ϕL_3 and $L_3\phi$, coincides with the domain $\mathfrak{D}$. *Thus, it follows that the commutator given by Eq.* (5.7.21) *is well-defined on the domain $\mathfrak{D}$, which is dense in $AC[0,2\pi)$.*

This is quite reassuring, but it is far from the end of the story! There are several problems yet to be discussed:

(a) The operator L_3, although indeed well-defined on $\mathfrak{D}$ and Hermitian on $\mathfrak{D}$, is *not* self-adjoint. This is important physically, since the spectral theorem (which validates quantum mechanical applications) applies *only* to self-adjoint operators (Reed and Simon [7], Weyl [8], von Neumann [9]).

(b) The domain $\mathfrak{D}$ is not physically satisfactory, since *the boundary condition* $[\phi(0)=\phi(2\pi)=0]$ *spoils the rotational symmetry of the physical problem.*

(c) The operator ϕ is also physically unsatisfactory, since it fails to be cyclic at the boundary.

Let us consider the problem of self-adjointness, item (a) above. The operator L_3 is sufficiently simple that one can determine the adjoint[1] L_3^* quite directly without using the complicated analysis developed by von Neumann for the general case.[2] Consider the defining relation for the adjoint operator L_3^*. For $f\in\mathfrak{D}(L_3)$ and $g\in\mathfrak{D}(L_3^*)$, we have (using an integration by parts)

$$(L_3 f,g)-(f,L_3^* g)=\bar{f}(2\pi)g(2\pi)-\bar{f}(0)g(0)\equiv 0.$$

Since $f\in\mathfrak{D}(L_3)$, it is required that $f(0)=f(2\pi)=0$, and hence the right-hand side vanishes *with no conditions on the function* g [aside from $g\in AC(0,2\pi)$]. It is clear that the domain of L_3^* is larger than (and includes) that of L_3; in symbols, $L_3\subset L_3^*$, as required by the general analysis.

In order to construct a self-adjoint extension of L_3, it is evident that one must weaken the boundary conditions on the admissible functions in such a way that the same conditions apply to both $\mathfrak{D}(L_3)$ and $\mathfrak{D}(L_3^*)$. There is a continuous family of such self-adjoint extensions, which we denote by $L_3^{(\alpha)}$. Then $L_3^{(\alpha)}$ is the operator $-i(d/d\phi)$ acting on the elements ψ of the domain $\mathfrak{D}_\alpha$ defined by

$$\mathfrak{D}_\alpha=\{\psi: \psi\in AC[0,2\pi), \psi(2\pi)=\alpha\psi(0), \alpha\in\mathbb{C}, |\alpha|=1\}. \tag{5.7.25}$$

[1]In this section only (and in related Note 2, p. 346), we follow the custom in mathematics and use the asterisk to denote the adjoint of an operator and the bar to denote complex conjugation.

[2]This discussion is taken from Reed and Simon [7, pp. 141ff]. This same example also appears in von Neumann [9].

Expressed in words: The self-adjoint operator $L_3^{(\alpha)}$ has the domain $\mathcal{D}_\alpha$, which is defined to be the space of absolutely continuous functions on the interval $[0,2\pi)$ satisfying the boundary condition $\psi(2\pi)=\alpha\psi(0)$, where α is a complex number of modulus 1. We note that $L_3 \subset L_3^{(\alpha)} = L_3^{(\alpha)*} \subset L_3^*$, as required by the general theory.

This result is satisfactory in that it validates the commutation relation, Eq. (5.7.21), not only for a *self-adjoint* operator but also on a *larger* space. The freedom to choose α is also satisfying, since this choice is a matter of physics, and it allows one to choose $\alpha=1$ as *the physical space of cyclic functions*, $f(\phi)=f(\phi+2\pi)$, *as required by rotational symmetry*. [Let us note that the other choices of α correspond each to a group contained in the covering group of the circle (the circle group having $\alpha=1$); this freedom results from the fact that the circle group is infinitely covered.]

This nicely disposes of items (a) and (b) above, but it *worsens* the problem presented by item (c), since the operator ϕ no longer leaves the domain $\mathcal{D}_\alpha$ invariant! Let us now turn to this problem, the task of finding a satisfactory definition of ϕ. It is not particularly helpful (although it is possible) to define (Nieto [19], Susskind and Glogower [20]) ϕ as a multiplication operator everywhere modulo 2π. Let us rather make use of the suggestion of Jordan [16] and consider, instead of ϕ, the operator $e^{i\phi}$. This disposes at once of the problem, since this operator is everywhere invariant under $\phi \to \phi + 2\pi$, and moreover leaves the domains $\mathcal{D}_\alpha$ invariant. The commutation relation then reads

$$[L_3, e^{i\phi}] = e^{i\phi}, \tag{5.7.26}$$

where we now understand the operator L_3 to be the self-adjoint operator denoted earlier by $L_3^{(\alpha=1)}$. This commutation relation is valid on the domain $\mathcal{D}_{\alpha=1}$.

We remark that this resolution of item (c) is not without objection, since technically the operator $e^{i\phi}$ is not a physical observable. We may, and shall, replace $e^{i\phi}$ by its physically observable components $\sin\phi$ and $\cos\phi$ where necessary, and regard this final difficulty more as an inconvenience than a genuine flaw.

Before developing the uncertainty relations that follow from Eq. (5.7.26), let us note explicitly the resolution of the "fallacy" with which we began. *The flaw lies in taking matrix elements of Eq.* (5.7.21) *using the eigenfunctions of* $L_3 = L_3^{(\alpha=1)}$, *since the operator* ϕ *takes the eigenfunctions out of the space*. Equivalently, the operator identity expressed by Eq. (5.7.21) is invalid when applied to an eigenfunction of L_3. This resolution of the problem is rather analogous to the resolution of a similar "fallacy" for the Heisenberg commutation relation itself. This latter fallacy results from taking matrix elements between "eigenvectors" of the position operator. The flaw here is

that the "eigenvectors" are *improper*, and not vectors in the Hilbert space. Thus, in both cases, the resolution of the "fallacy" is to deny the validity of the commutation relation when acting on eigenvectors.

Let us turn now to the uncertainty relations. It is convenient to consider first the operators $\sin\phi$ and $\cos\phi$ defined by

$$\sin\phi \equiv (e^{i\phi} - e^{-i\phi})/2i,$$
$$\cos\phi \equiv (e^{i\phi} + e^{-i\phi})/2, \tag{5.7.27}$$

for which the commutation relations are

$$[L_3, \sin\phi] = -i\cos\phi,$$
$$[L_3, \cos\phi] = i\sin\phi. \tag{5.7.28}$$

Consider the first commutator in Eqs. (5.7.28). Let us define the operators L and X by

$$L \equiv L_3 - \langle L_3 \rangle,$$
$$X \equiv \sin\phi - \langle \sin\phi \rangle. \tag{5.7.29}$$

Using $[L, X] = -i\cos\phi$, we may now repeat the procedure given by Eqs. (5.7.7)–(5.7.14). We find that the minimum uncertainty relation

$$\Delta L_3 \Delta(\sin\phi) = \tfrac{1}{2}|\langle\cos\phi\rangle| \tag{5.7.30}$$

will hold if there exists a (minimum uncertainty) state $|\psi\rangle$, which satisfies the following conditions:[1]

$$L|\psi\rangle = \lambda X|\psi\rangle,$$
$$(\Delta L_3)^2 = -i\lambda\langle\cos\phi\rangle/2,$$
$$(\Delta(\sin\phi))^2 = i\langle\cos\phi\rangle/2\lambda. \tag{5.7.31}$$

Using $L_3 = -i\partial/\partial\phi$, we may integrate the first equation in (5.7.31) to obtain the following relation that the minimum uncertainty state $\psi(\phi) = \langle\phi|\psi\rangle$ must satisfy:

$$\psi(\phi) = N^{-\frac{1}{2}} \exp i[\langle L_3\rangle\phi - \lambda\cos\phi - \lambda\langle\sin\phi\rangle\phi], \tag{5.7.32}$$

where N is a normalization factor.

[1] One must be particularly cautious with the (standard) notation used here. For example, $\langle\cos\phi\rangle$ is *not* a function of ϕ, but rather a complex number whose value depends on the state $|\psi\rangle$—namely, $\langle\cos\phi\rangle \equiv \langle\psi|\cos\phi|\psi\rangle = \alpha(\psi) =$ complex number.

The requirement that $\psi(\phi+2\pi)=\psi(\phi)$ forces the relation

$$\exp i2\pi[\langle L_3\rangle-\lambda\langle\sin\phi\rangle]=1;$$

that is,

$$\langle L_3\rangle-\lambda\langle\sin\phi\rangle=m, \tag{5.7.33}$$

where $m=0,\pm1,\pm2,\ldots$. Thus, we find a denumerable infinity of minimum uncertainty states of the form

$$\psi_m(\phi)=N^{-\frac{1}{2}}e^{i(m\phi-\lambda\cos\phi)}, \qquad m=0,\pm1,\pm2,\ldots, \tag{5.7.34}$$

where we note from Eqs. (5.7.31) that λ is a pure imaginary number of the form $\lambda=i\mu$ with $\mu>0$ (see footnote, p. 311).

Let us digress a moment to give several integrals that are useful for interpreting the results given by Eqs. (5.7.30)–(5.7.34):

$$\begin{aligned}2\pi I_0(2\mu)&=\int_0^{2\pi}d\phi\, e^{2\mu\cos\phi},\\ 2\pi I_1(2\mu)&=\int_0^{2\pi}d\phi\, e^{2\mu\cos\phi}\cos\phi=2\mu\int_0^{2\pi}d\phi\, e^{2\mu\cos\phi}\sin^2\phi,\\ 0&=\int_0^{2\pi}d\phi\, e^{2\mu\cos\phi}\sin\phi.\end{aligned} \tag{5.7.35}$$

In these results, I_n is a modified Bessel function (Watson [21]), and μ is an arbitrary real parameter.

We next introduce $\lambda=i\mu$ explicitly into Eq. (5.7.34) and define

$$\psi_{m\mu}(\phi)=N^{-\frac{1}{2}}e^{im\phi+\mu\cos\phi} \tag{5.7.36}$$

for all $m=0,\pm1,\pm2,\ldots$ and for all positive values of μ.

The results given by Eqs. (5.7.35) now allow us to give explicitly the normalization of the states (5.7.36) and the expectation values of $\cos\phi$, $\sin^2\phi$, $\sin\phi$, and L_3 for these states:

$$\begin{aligned}N&=2\pi I_0(2\mu),\\ \langle\cos\phi\rangle&=2\mu\langle\sin^2\phi\rangle=I_1(2\mu)/I_0(2\mu),\\ \langle\sin\phi\rangle&=0,\\ \langle L_3\rangle&=m.\end{aligned} \tag{5.7.37}$$

We conclude: The states in the set $\{|\psi_{m\mu}\rangle: m=0,\pm1,\pm2,\ldots;\mu>0\}$ are minimum uncertainty states for the observables L_3 and $\sin\phi$. The dispersions of these observables in the state $|\psi_{m\mu}\rangle$ are given by

$$\begin{aligned}(\Delta L_3)^2&=\mu\langle\cos\phi\rangle/2,\\ (\Delta(\sin\phi))^2&=\langle\sin^2\phi\rangle=\langle\cos\phi\rangle/2\mu\end{aligned} \tag{5.7.38}$$

and satisfy the minimum uncertainty relations

$$\Delta L_3\Delta(\sin\phi)=\tfrac{1}{2}|\langle\cos\phi\rangle| \tag{5.7.39}$$

associated with the commutation relation

$$[L_3,\sin\phi]=-i\cos\phi. \tag{5.7.40}$$

These dispersions also satisfy the relation

$$\Delta L_3/\Delta(\sin\phi)=\mu. \tag{5.7.41}$$

The explicit dependence of ΔL_3 and $\Delta(\sin\phi)$ on μ is determined by Eqs. (5.7.38) and

$$\langle\cos\phi\rangle=I_1(2\mu)/I_0(2\mu). \tag{5.7.42}$$

A similar result to that stated above also holds for the observables L_3 and $\cos\phi$. [Interchange $\cos\phi$ and $\sin\phi$ in all of Eqs. (5.7.36)–(5.7.42), replacing also i by $-i$ in Eq. (5.7.40)].

Remarks. (*a*) The only parameter in the uncertainty relation (5.7.39) is μ. For large μ one may show, using asymptotic properties of the modified Bessel functions, that $\langle\cos\phi\rangle\sim1-1/4\mu$. In this case we see from Eqs. (5.7.38) that ΔL_3 is large and $\Delta(\sin\phi)$ is small. Since $\langle\sin\phi\rangle=0$ and $\langle\cos\phi\rangle\cong1$, the uncertainty $\Delta\phi$ in ϕ is small, and we recover the usual uncertainty relation result that ΔL_3 is large when $\Delta\phi$ is small.

(*b*) For the opposite extreme—μ small—we find that $\langle\cos\phi\rangle\sim\mu$, $(\Delta L_3)\sim\mu/\sqrt{2}$, and $\Delta(\sin\phi)\sim1/\sqrt{2}$. Thus, in the limit $\mu\to0$, the uncertainty relation (5.7.39) becomes trivial (both sides zero), thereby escaping the dilemma ($\Delta\phi$ undefined) of the incorrect result based naively on Eq. (5.7.21).

Appendix to Section 3. *Quantum mechanics of discrete rotations.* Weyl, in his discussion of quantum kinematics (Ref. [8, pp. 272–276)], recognized that the Heisenberg commutation relations were a particular instance of a much more basic view:[1]

> We have thus found a very natural interpretation of quantum kinematics as described by the commutation rules. *The kinematical structure of a physical system is expressed by an irreducible Abelian group of unitary ray rotations in system space. The real elements of the algebra of this group are the physical quantities of the system; the representation of the abstract group by rotations of system space associates with each such quantity a definite Hermitian form which "represents" it.* If the group is *continuous* this procedure automatically leads to *Heisenberg's* formulation; in particular, we have seen how the *pairs of canonical variables* then result from the requirement of irreducibility, whence the number of parameters in such an irreducible Abelian group must be *even*."

Weyl recognized further that "our general principle allows for the possibility that the Abelian rotation group is entirely discontinuous, or that it may even be a finite group."

In our opinion, the full import of this insight has yet to be obtained. As an illustration, we shall consider the example of discrete rotations (that is, a finite group) in some detail.

Let U and V represent a pair of canonical elements corresponding to the finite group analog to Eq. (5.7.5). Thus, U and V are unitary transformations of a finite-dimensional Hilbert space that satisfy the relation

$$UV=\varepsilon VU,$$

where $\varepsilon=\exp(2\pi i/N)$ is an Nth root of unity for some integer N. Thus, one has

$$U^l V^k = e^{2\pi i k l/N} V^k U^l.$$

(In the applications to angular momentum to be discussed in Section 4, the integer N will be taken to be equal to $2j+1$.)

In his discussion of such discrete Weyl systems, Schwinger[2] [22] found it advantageous to introduce the (bra-vector) orthonormal basis

$$\{\langle a^k| : k=1,2,\ldots,N\},$$

where

$$\langle a^k|V=\langle a^{k+1}|, \qquad k=1,2,\ldots,N,$$

with $\langle a^{N+1}|=\langle a^1|$. Thus, two bra-vectors, $|a^{k'}\rangle$ and $|a^k\rangle$, are equal if $k'\equiv k(\text{mod } N)$.

[1]The italics in the quotation are in the original text.

[2]See also the references in [22] to Schwinger's papers in the *Proceedings of the National Academy of Sciences.* We follow Schwinger's notation and presentation of the discrete Weyl systems.

Repetition of the action of the unitary operator V defines a sequence of linearly independent unitary operators, $V, V^2, V^3\ldots$ with the actions given by

$$\langle a^k|V^n=\langle a^{k+n}|, \qquad n=1,2,\ldots,$$

until one arrives at

$$\langle a^k|V^N=\langle a^{k+N}|=\langle a^k|, \qquad \text{each } k=1,2,\ldots,N.$$

Thus, V^N is the unit operator,

$$V^N=1.$$

This result shows that the eigenvalues of the operator V are the N distinct complex roots of unity:

$$\{v^k : v=e^{2\pi i/N} \text{ and } k=0,\ldots,N-1\}.$$

We may now factor the equation $V^N-1=0$ in the form

$$[(V/v)-1]\sum_{k=0}^{N-1}(V/v)^k=0.$$

This result, in turn, shows that the expression for the projection operator $P(v^k)$ for the eigenspace corresponding to the eigenvalue v^k is

$$P(v^k)=N^{-1}\sum_{l=0}^{N-1}e^{-2\pi i k l/N}V^l.$$

Letting $|v^k\rangle$ denote the eigenvector of V corresponding to the eigenvalue v^k, we may also express the projection operator $P(v^k)$ as

$$P(v^k)=|v^k\rangle\langle v^k|.$$

Applying the operator $P(v^k)$ to the basis $\{\langle a^n|\}$, we thus find

$$\begin{aligned}\langle a^n|P(v^k)&=\langle a^n|v^k\rangle\langle v^k|,\\ &=N^{-1}\sum_{l=0}^{N-1}e^{-2\pi i k l/N}\langle a^n|V^l,\\ &=N^{-1}\sum_{l=0}^{N-1}e^{-2\pi i k l/N}\langle a^{n+l}|.\end{aligned}$$

These results determine the transformation coefficients between the two bases, which, with a particular phase choice, read

$$\langle a^k | v^l \rangle = N^{-\frac{1}{2}} e^{2\pi i k l/N},$$
$$\langle v^l | a^k \rangle = N^{-\frac{1}{2}} e^{-2\pi i k l/N}.$$

This is equivalent to expressing the eigen-bras of V in terms of the original basis $\{|\langle a^l|\}$ as given by

$$\langle v^k | = N^{-\frac{1}{2}} \sum_{l=0}^{N-1} \langle a^l | e^{-2\pi i k l/N}.$$

The operator U—in the canonical pair U, V—has the effect of cyclically permuting the basis $\{\langle v^k|\}$, in a way similar to the action of V on the basis $\{\langle a^l|\}$. In particular, one has

$$\langle v^k | U = \langle v^{k-1} |,$$

which, in turn, implies that $U^N = 1$, in accord with

$$U^l V^k = e^{2\pi i k l/N} V^k U^l.$$

The eigenvalues of the operator U are the Nth roots of unity, just as for the operator V. In fact, the vectors in basis $\{\langle a^k|\}$ can be seen to be precisely the eigenvectors of U, so that

$$\langle a^k | = \langle u^k |.$$

The complementary pair of operators, U and V, are the generators of a complete orthonormal operator basis for the set of all operators mapping the space spanned by $\{\langle a^n|\}$ into itself. This operator basis may be taken to be the set of N^2 operators given by

$$X(m,n) = N^{-\frac{1}{2}} U^m V^n, \qquad m, n = 0, 1, \ldots, N-1.$$

The orthonormality and completeness (which may be shown by Schur's lemma) are expressed by the relations

$$\langle X^\dagger(m,n) | X(m',n') \rangle = \delta_{mm'} \delta_{nn'}$$

for $m, m', n, n' = 0, \ldots, N-1$.

Remarks. (*a*) Schwinger has shown that a kind of ergodic theorem is valid for this operator system, in which an average over spectral translations is equated to an average over states.

(*b*) The importance of these discrete Weyl systems lies in the fact that the pair of operators U, V generates a complete operator basis (for any N), and the two operators are maximally incompatible, as expressed by the aspect of complementarity.

(*c*) In the limit where $N \to \infty$, one recovers the Weyl representation theorem for operators of the Hilbert–Schmidt class. Thus, one finds that

$$A = \sum_{m,n} a(m,n) X(m,n),$$
$$\operatorname{tr}(A^\dagger A) = \sum_{m,n} |a(m,n)|^2$$

becomes Weyl's result in the limit:

$$A = \iint_{-\infty}^{\infty} d\sigma\, d\tau\, a(\sigma,\tau) e^{i(\sigma q - \tau p)},$$
$$\operatorname{tr}(A^\dagger A) = \iint_{-\infty}^{\infty} d\sigma\, d\tau |a(\sigma,\tau)|^2,$$

where p, q denote a canonical pair of momentum and position operators.

(*d*) One can extend this construction to products of Weyl systems defined over the prime numbers and to the infinite prime (thereby obtaining Schwinger's "special canonical group," which is an alternative approach to Dirac's delta function). There would seem to be many interesting generalizations of the Weyl system to, say, idele and adele groups and the like.[1]

Let us conclude by noting that minimum uncertainty states do exist in discrete Weyl systems, although, for reasons of brevity, we shall not give these states here. (It should also be noted that this basic structural insight of Weyl has been extensively developed for continuous systems. A guide to this literature may be found in Grossmann [24a] and Wolf [24b].)

4. Uncertainty Relations for Angular Momentum in Three-Dimensional Space

Introductory remarks. The first question that must be settled before the desired uncertainty relations can be obtained is this: What is the canonical set of variables for the discussion of angular momentum? We have, of

[1] Segal [23] discusses some of these possibilities briefly; Mackey [24, pp. 52–53] also mentions such possible generalizations.