The intelligent states. II. The computation of the Clebsch–Gordan coefficients

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(Received 8 August 1977)

In this second of a series of papers on quasi-intelligent states, we give a general method for the computation of the Clebsch–Gordan coefficients for these states. In a special case, these coefficients are found to be closely related to the Clebsch–Gordan coefficients of the rotation group. We also discuss the nonuniqueness resulting naturally from the overcompleteness of these states.

1. INTRODUCTION

In the first paper of this series[1] (this paper will henceforth be referred to as I) we introduced the group-theoretic formulation for the study of the quasi-intelligent states which are generalizations of the states (called the intelligent states) satisfying equality in the Heisenberg uncertainty relation $\Delta J_1^2 \Delta J_2^2 \geq \frac{1}{4}|\langle J_3\rangle|^2$. In particular, we presented a method based on the knowledge of a certain generating function for the computation of matrix elements of polynomials in the infinitesimal generators of rotations in three dimensions between quasi-intelligent states.

In this paper, we continue this study and compute the Clebsch—Gordan coefficients for these states. Not surprisingly, these come out to be very closely related to the Clebsch—Gordan coefficients of the rotation group.

The present paper is organized as follows.

In Sec. 2, after redefining, for completeness, the various operators we compute their effect on the quasi-intelligent states. We utilize the results of this section in the next section to show that any Wigner state can be expressed as a linear combination of the quasi-intelligent states for any given complex number $\alpha \neq \pm 1$.[2] This is effectively the inversion of the expression for a quasi-intelligent state in terms of the Wigner states which was given in I. Indeed in Appendix A, we verify the correctness of this inversion. In Sec. 4, we derive the Clebsch—Gordan coefficients for the quasi-intelligent states and show that, up to a normalization independent of the magnetic quantum numbers, these Clebsch—Gordan coefficients for the same α are very closely related to the Clebsch—Gordan (CG) coefficients of the rotation group. In Appendix B, we show that this, so far, unknown normalization coefficient is indeed 1.

We emphasize that the quasi-intelligent states are eigenstates of a non-Hermitian operator having the same finite spectrum as the operator J_3 for a given angular momentum j. The non-Hermiticity of this operator makes the quasi-intelligent states nonorthogonal, thus some steps in the computation of the CG coefficients have to be handled rather carefully.

In Appendix C, we exemplify a consequence of the nonorthogonality of the quasi-intelligent states by showing that we *only* obtain a generalization of the expansion of the unit operator commonly known as completeness. Thus the quasi-intelligent states are, perhaps, not complete in this sense, though they are definitely complete in the sense that any Wigner state can be expressed as a linear combination of them.

2. THE OPERATORS $J_3'(\alpha)$, $J_\pm'(\alpha)$ AND THEIR EFFECT ON A QUASI-INTELLIGENT STATE

As in I, the *normalized* quasi-intelligent states corresponding to a given angular momentum j and a complex number $\alpha \neq \pm 1$ are given by[3]

$$|jm\alpha\rangle = [a^j{}_m(\alpha)]^{-1} \exp(\theta J_3) \exp(-i\tfrac{1}{2}\pi J_2)\,|jm\rangle, \tag{1}$$

where

$$e^\theta = \left(\frac{1-\alpha}{1+\alpha}\right)^{1/2} \tag{2}$$

and

$$a^j{}_m(\alpha) = \{\langle jm|\exp[-(\theta+\theta^*)J_1]\,|jm\rangle\}^{1/2}. \tag{3}$$

These states are eigenstates of the operator $J_3'(\alpha) = (J_1 - i\alpha J_2)/(1-\alpha^2)^{1/2}$. Indeed

$$J_3'(\alpha)\,|jm\alpha\rangle = m\,|jm\alpha\rangle. \tag{4}$$

Thus for a given j and a complex number $\alpha \neq \pm 1$, $J_3'(\alpha)$ has the spectrum $-j \leq m \leq j$, exactly the same as of the operator J_3.[4]

The basic difficulty in handling the quasi-intelligent states is the result of the obvious non-Hermiticity of the operator $J_3'(\alpha)$. As an immediate consequence, the corresponding eigenstates $|jm\alpha\rangle$ *might* not be orthogonal. Indeed it was explicitly verified in I that these states are not orthgonal for real $\theta \neq 0$. We also showed in I that for real $\theta \neq 0$, the normalization coefficients $a^j{}_m(\alpha) \neq 1$.

We define the operators

$$J_\pm'(\alpha) = \mp\frac{\alpha}{(1-\alpha^2)^{1/2}} J_1 \pm \frac{i}{(1-\alpha^2)^{1/2}} J_2 - J_3, \tag{5}$$

which together with $J_3'(\alpha)$ defined above satisfy the commutation relations

$$[J_3'(\alpha), J_\pm'(\alpha)] = \pm J_\pm'(\alpha), \tag{6}$$

$$[J_+'(\alpha), J_-'(\alpha)] = 2J_3'(\alpha), \tag{7}$$

which are exactly the same as those satisfied by J_3, $J_\pm = J_1 \pm iJ_2$.

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 0022-2488/78/1906-1397$1.00

$$\mathbf{J}^2 = J_1^2 + J_2^2 + J_3^2 = \tfrac{1}{2}[J_+J_- + J_-J_+] + J_3^2$$
$$= \tfrac{1}{2}[J'_+(\alpha)J'_-(\alpha) + J'_-(\alpha)J'_+(\alpha)] + J_3'^2(\alpha). \tag{8}$$

Now we wish to compute the effect of $J'_\pm(\alpha)$ on $|jm\alpha\rangle$. This can be immediately known from

$$\exp(-\theta J_3)J'_\pm(\alpha)\exp(\theta J_3) = \pm iJ_2 - J_3 \tag{9a}$$

and

$$\exp(i\tfrac{1}{2}\pi J_2)(\pm iJ_2 - J_3)\exp(-i\tfrac{1}{2}\pi J_2) = J_1 \pm iJ_2 = J_\pm. \tag{9b}$$

Thus,[5]

$$J'_\pm(\alpha)|jm\alpha\rangle = [a^j{}_m(\alpha)]^{-1}J'_\pm(\alpha)\exp(\theta J_3)\exp(-i\tfrac{1}{2}\pi J_2)|jm\rangle$$
$$= [a^j{}_m(\alpha)]^{-1}\exp(\theta J_3)\exp(-i\tfrac{1}{2}\pi J_2)J_\pm|jm\rangle$$
$$= \frac{a^j{}_{m\pm1}(\alpha)}{a^j{}_m(\alpha)}[(j\mp m)(j\pm m+1)]^{1/2}|j(m\pm1)\rangle. \tag{10}$$

We can immediately verify that Eqs. (4) and (10) are consistent with Eqs. (6)–(8).

In the following, we shall also require use of the states $|jm\alpha^c\rangle$. Here α^c is defined such that $\alpha \to -\alpha^*$, whereas $(1-\alpha^2)^{1/2} \to [(1-\alpha^2)^{1/2}]^*$ or in the language of θ, $\theta \to -\theta^*$. Using the variable τ, this operation[6] is expressed as $\tau \to (\tau^*)^{-1}$. The state $|jm\alpha\rangle$ is orthogonal to $|j'm'\alpha^c\rangle$. Indeed,

$$\langle j'm'\alpha^c|jm\alpha\rangle = \delta_{jj'}\delta_{mm'}[a^j{}_m(\alpha)]^{-2} \tag{11a}$$

since

$$a^j{}_m(\alpha) = a^j{}_m(\alpha^c), \tag{11b}$$

as can be deduced from Eq. (3) above or one may see it manifestly in Eq. (30b) in I.

3. EXPRESSION FOR A WIGNER STATE IN TERMS OF THE QUASI-INTELLIGENT STATES

In I, we derived the manifest expressions[7]

$$|jm\alpha\rangle = [a^j{}_m(\alpha)]^{-1}\left(\frac{(j-m)!}{(j+m)!}\right)^{1/2}\sum_{m_1,r}|jm_1\rangle\exp(m_1\theta)$$
$$\times\left(\frac{(j+m_1)!}{(j-m_1)!}\right)^{1/2}2^{-j+r}(-1)^r$$
$$\times\frac{(2j-r)!}{r!(j-m-r)!(j+m_1-r)!} \tag{12a}$$

$$= [a^j{}_m(\alpha)]^{-1}\left(\frac{(j+m)!}{(j-m)!}\right)^{1/2}\sum_{m_1,r}|jm_1\rangle\exp(m_1\theta)$$
$$\times\left(\frac{(j-m_1)!}{(j+m_1)!}\right)^{1/2}2^{-j+r}(-1)^{m_1-m+r}$$
$$\times\frac{(2j-r)!}{r!(j+m-r)!(j-m_1-r)!}, \tag{12b}$$

which are equivalent to the concise Eq. (1) in terms of the operation of the infinitesimal generators of the rotation group. Our purpose, in this section, is to utilize the results of the previous section to invert this equation to obtain an expression for any Wigner state as a linear combination of the quasi-intelligent states for any given complex $\alpha \neq \pm 1$.

To achieve it, let us go back to Eq. (1) which results in

$$|jm\rangle = a^j{}_m(\alpha)\exp(i\tfrac{1}{2}\pi J_2)\exp(-\theta J_3)|jm\alpha\rangle, \tag{13}$$

which is a *nonmanifest* form of the inversion we are seeking. Since we do know how $J'_3(\alpha)$, $J'_\pm(\alpha)$ operate on the quasi-intelligent states $|jm\alpha\rangle$, we attempt to express the operator $\exp(i\tfrac{1}{2}\pi J_2)\exp(-\theta J_3)$ in the form $\exp[aJ'_-(\alpha)]\exp[bJ'_3(\alpha)]\exp[cJ'_+(\alpha)]$. This can be done using the 2×2 representation $\sigma_i/2$ of the operators J_i. We find

$$a = -1, \quad e^b = \frac{1}{2\tau}, \quad c = \tau,$$

or

$$\exp\ i\frac{\pi}{2}J_2\ \ \exp(-\theta J_3) = \exp[-J'_-(\alpha)]\left(\frac{1}{2\tau}\right)^{J'_3(\alpha)}$$
$$\times\exp[\tau J'_+(\alpha)]. \tag{14}$$

Thus

$$|jm\rangle = a^j{}_m(\alpha)\exp[-J'_-(\alpha)]\left(\frac{1}{2\tau}\right)^{J'_3(\alpha)}\exp[\tau J'_+(\alpha)]|jm\alpha\rangle, \tag{15}$$

where for the operators on the right, we have already understood their operation on $|jm\alpha\rangle$ in the previous section. Utilizing Eqs. (4) and (10), we arrive at

$$|jm\rangle = \exp(-m\theta)\left(\frac{(j-m)!}{(j+m)!}\right)^{1/2}\sum_{m'm''}|jm''\alpha\rangle\left(\frac{(j-m'')!}{(j+m'')!}\right)^{1/2}$$
$$\times\alpha^j{}_{m''}(\alpha)(-1)^{m'-m''}2^{-m'}\frac{(j+m')!}{(j-m')!(m'-m)!(m'-m'')!}. \tag{16}$$

In Appendix A, we shall explicitly verify the correctness of the above inversion.

In Eq. (16), we have expressed a given Wigner state $|jm\rangle$ as a linear combination of the quasi-intelligent states $|jm\alpha\rangle$ for a given complex $\alpha \neq \pm 1$ and $-j \leq m \leq j$. This shows that the quasi-intelligent states are indeed complete in the sense that any Wigner state can be expressed as a linear combination of them.

4. THE CLEBSCH–GORDAN COEFFICIENTS FOR THE QUASI-INTELLIGENT STATES

Now we have all the machinery at our disposal to enable us to compute the Clebsch–Gordan coefficients for the quasi-intelligent states. These coefficients are defined through the equation

$$|j_1m_1\alpha_1\rangle|j_2m_2\alpha_2\rangle$$
$$= \sum_{jm}(jm\alpha|j_1m_1\alpha_1;j_2m_2\alpha_2)|jm\alpha\rangle, \tag{17a}$$

where we have used *round brackets* to distinguish them from the usual Clebsch–Gordan coefficients of the rotation group. Equation (17) above expresses the completeness of the states $|jm\alpha\rangle$ for any complex $\alpha \neq \pm 1$ in the sense that any Wigner state $|jm\rangle$ can be expressed as a linear combination of them. Note particularly that, *in general*, the sum in Eq. (17) above is over both j and m where $|j_1 - j_2| \leq j \leq j_1 + j_2$ and $-j \leq m \leq j$ for a given j.

The Clebsch–Gordan coefficient $(jm\alpha|j_1m_1\alpha_1;j_2m_2\alpha_2)$ can be expressed as an inner product using the states $|jm\alpha^c\rangle$. Indeed

$$(jm\alpha|j_1m_1\alpha_1;j_2m_2\alpha_2)$$
$$=\langle jm\alpha^c|(|j_1m_1\alpha_1\rangle|j_2m_2\alpha_2\rangle)\rangle[a^j{}_m(\alpha)]^2 \tag{17b}$$

on making use of Eq. (11a). The inner product on the right above can be computed using Eqs. (12) and hence we can obtain a *general* Clebsch–Gordan coefficient for different values of α_1, α_2, and α. This coefficient is, however, very complicated and *cannot* be simplified except in special cases. In the following, we discuss a rather special case where $\alpha_1=\alpha_2=\alpha$. Now the definition of α^c has been chosen such that from

$$J_3'(\alpha)|jm\alpha\rangle=m|jm\alpha\rangle$$

one concludes

$$\langle jm\alpha^c|J_3'(\alpha)=m\langle jm\alpha^c|.$$

Thus Eq. (17b) implies, for the special case when $\alpha=\alpha_1=\alpha_2$,

$$(jm\alpha|j_1m_1\alpha_1;j_2m_2\alpha_2)$$
$$=\delta_{m,m_1+m_2}(jm\alpha|j_1m_1\alpha_1;j_2m_2\alpha_2), \tag{18}$$

or the summation in Eq. (17a) over m can be omitted with the understanding that $m=m_1+m_2$, i.e.,

$$|j_1m_1\alpha_1\rangle|j_2m_2\alpha_2\rangle$$
$$=\sum_j(j(m_1+m_2)\alpha|j_1m_1\alpha_1;j_2m_2\alpha_2)|jm\alpha\rangle. \tag{19}$$

Note that since the α's are kept the same throughout

$$J_3'^{(1)}(\alpha)+J_3'^{(2)}(\alpha)=\frac{J_1^{(1)}-i\alpha J_2^{(1)}}{(1-\alpha^2)^{1/2}}+\frac{J_1^{(2)}-i\alpha J_2^{(2)}}{(1-\alpha^2)^{1/2}}$$
$$=\frac{(J_1^{(1)}+J_1^{(2)})-i\alpha(J_2^{(1)}+J_2^{(2)})}{(1-\alpha^2)^{1/2}}$$
$$=\frac{J_1-i\alpha J_2}{(1-\alpha^2)^{1/2}}=J_3'(\alpha),$$

which results in the simplification given in Eq. (18) above and the ones which follow. Similar results hold for the operators $J_\pm'(\alpha)$.

Next we operate both sides of Eq. (19) by $J_+'(\alpha)=J_+'^{(1)}(\alpha)+J_+'^{(2)}(\alpha)$. Using Eq. (10), this operation gives

$$\frac{a^{j_1}{}_{(m_1+1)}(\alpha)}{a^{j_1}{}_{m_1}(\alpha)}[(j_1-m_1)(j_1+m_1+1)]^{1/2}|j_1(m_1+1)\alpha\rangle|j_2m_2\alpha\rangle$$
$$+\frac{a^{j_2}{}_{(m_2+1)}(\alpha)}{a^{j_2}{}_{m_2}(\alpha)}[(j_2-m_2)(j_2+m_2+1)]^{1/2}$$
$$\times|j_1m_1\alpha\rangle|j_2(m_2+1)\alpha\rangle$$
$$=\sum_j\frac{a^j{}_{(m+1)}(\alpha)}{a^j{}_m(\alpha)}[(j-m)(j+m+1)]^{1/2}$$
$$\times(jm\alpha|j_1m_1\alpha;j_2m_2\alpha)|j(m+1)\alpha\rangle.$$

Next we use Eq. (17) again for $|j_1(m_1+1)\alpha\rangle|j_2m_2\alpha\rangle$ and $|j_1m_1\alpha\rangle|j_2(m_2+1)\alpha\rangle$ which appear on the left-hand side of the above equation. This results in

$$\frac{a^{j_1}{}_{(m_1+1)}(\alpha)}{a^{j_1}{}_{m_1}(\alpha)}[(j_1-m_1)(j_1+m_1+1)]^{1/2}$$
$$\times\sum_j(j(m+1)\alpha|j_1(m_1+1)\alpha;j_2m_2\alpha)|j(m+1)\alpha\rangle$$
$$+\frac{a^{j_2}{}_{(m_2+1)}(\alpha)}{a^{j_2}{}_{m_2}(\alpha)}[(j_2-m_2)(j_2+m_2+1)]^{1/2}$$
$$\times\sum_j(j(m+1)\alpha|j_1m_1\alpha;j_2(m_2+1)\alpha)|j(m+1)\alpha\rangle$$
$$=\sum_j\frac{a^j{}_{(m+1)}(\alpha)}{a^j{}_m(\alpha)}[(j-m)(j+m+1)]^{1/2}$$
$$\times(jm\alpha|j_1m_1\alpha;j_2m_2\alpha)|j(m+1)\alpha\rangle. \tag{20}$$

In order to remove the sum over j we take the inner product of both sides with $\langle j'(m+1)\alpha|$ and use $\langle j'(m+1)\alpha|j(m+1)\alpha\rangle=\delta_{jj'}$. [Note that this orthonormality involves j and not m and hence does not depend upon the non-Hermiticity of $J_3'(\alpha)$.] We also multiply by $a^{j_1}{}_{m_1}(\alpha)a^{j_2}{}_{m_2}(\alpha)[a^j{}_{(m+1)}(\alpha)]^{-1}$ to arrive at

$$\frac{a^{j_1}{}_{(m_1+1)}(\alpha)a^{j_2}{}_{m_2}(\alpha)}{a^j{}_{(m+1)}(\alpha)}[(j_1-m_1)(j_1+m_1+1)]^{1/2}$$
$$\times(j(m+1)\alpha|j_1(m_1+1)\alpha;j_2m_2\alpha)$$
$$+\frac{a^{j_1}{}_{m_1}(\alpha)a^{j_2}{}_{(m_2+1)}(\alpha)}{a^j{}_{m+1}(\alpha)}[(j_2-m_2)(j_2+m_2+1)]^{1/2}$$
$$\times(j(m+1)\alpha|j_1m_1\alpha;j_2(m_2+1)\alpha)$$
$$=\frac{a^{j_1}{}_{m_1}(\alpha)a^{j_2}{}_{m_2}(\alpha)}{a^j{}_m(\alpha)}[(j-m)(j+m+1)]^{1/2}$$
$$\times(jm\alpha|j_1m_1\alpha;j_2m_2\alpha), \tag{21}$$

which shows that the quantities

$$\frac{a^{j_1}{}_{m_1}(\alpha)a^{j_2}{}_{m_2}(\alpha)}{a^j{}_m(\alpha)}(jm\alpha|j_1m_1\alpha;j_2m_2\alpha)$$

satisfy the same recursion relation as the one satisfied by the CG coefficients $\langle jm|j_1m_1;j_2m_2\rangle$ of the rotation group. Hence we conclude that

$$(jm\alpha|j_1m_1\alpha;j_2m_2\alpha)$$
$$=\beta(j_1j_2j;\alpha)\frac{a^j{}_m(\alpha)}{a^{j_1}{}_{m_1}(\alpha)a^{j_2}{}_{m_2}(\alpha)}\langle jm|j_1m_1;j_2m_2\rangle, \tag{22}$$

where the coefficient $\beta(j_1j_2j;\alpha)$ will have to be fixed by normalization and choice of phase.[8] These coefficients are independent of the magnetic quantum numbers. Indeed, in Eq. (22) we have been able to separate the dependence of the CG coefficient for the intelligent states into the corresponding CG coefficient of the rotation group and the normalization factors of the involved intelligent states.

To calculate the β's, it is clear from the above equation that if we could obtain the coefficient on the left for some special values of the magnetic quantum numbers, we would be able to obtain the β in this equation. Note that Eq. (19) implies that the CG coefficient $(jm\alpha|j_1m_1\alpha;j_2m_2\alpha)$ can be obtained by taking the inner product of $|j_1m_1\alpha\rangle|j_2m_2\alpha\rangle$ with $\langle jm\alpha|$. In Appendix B we shall carry out this program and show that the β's can, in fact, be chosen to be just one. Thus we find finally

$$(jm\alpha|j_1m_1\alpha;j_2m_2\alpha)$$
$$=\frac{a^j{}_m(\alpha)}{a^{j_1}{}_{m_1}(\alpha)a^{j_2}{}_{m_2}(\alpha)}\langle jm|j_1m_1;j_2m_2\rangle, \tag{23}$$

where both sides are identically zero if $m\neq m_1+m_2$.

At this stage, we wish to remark on the possible nonuniqueness in the expansion for the product

$|j_1m_1\alpha_1\rangle|j_2m_2\alpha_2\rangle$ in terms of the states $|jm\alpha\rangle$. Already in Eq. (17a) wherein we defined the Clebsch–Gordan coefficients, we have the built-in nonuniqueness since the complex "α" appearing on the right is at our disposal. Note that the states $|jm\alpha\rangle$ are complete for each $\alpha \neq 1$. Considering the set of all "α" at our disposal, we possess a highly *overcomplete* set of vectors which should naturally result in the nonuniqueness expressed above. In the special case expressed in Eq. (19), we have restricted ourselves to $\alpha=\alpha_1$ (note that $\alpha_1=\alpha_2$). Using the states $|jm\alpha'\rangle$ and Eq. (III.6) in Appendix C we can rewrite Eq. (19) as

$$|j_1m_1\alpha\rangle|j_2m_2\alpha\rangle = \sum_{jm}(j(m_1+m_2)\alpha|j_1m_1\alpha;j_2m_2\alpha) \times\langle jm\alpha'^c|j(m_1+m_2)\alpha\rangle[a^j{}_m(\alpha')]^2|jm\alpha'\rangle, \tag{24}$$

which is an expansion as a linear combination of $|jm\alpha'\rangle$ and reduces to Eq. (19) in case $\alpha=\alpha'$ on using Eq. (11a). Note that the expression on the right in the above equation has the additional (perhaps artificial in this special case) summation over m.

ACKNOWLEDGMENTS

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

APPENDIX A: EXPLICIT VERIFICATION OF THE INVERSION IN EQ. (16)

We proved Eq. (16) in Sec. 3 by using the knowledge of the effect of the operators $J'_\pm(\alpha)$ on $|jm\alpha\rangle$. Now we verify that Eq. (16) indeed provides an inversion to Eq. (12). For this purpose, we substitute for $|jm''\alpha\rangle$ in Eq. (16) from Eq. (12). This makes the expression E on the right-hand side of Eq. (16) become

$$E=\exp(-m\theta)\left[\frac{(j-m)!}{(j+m)!}\right]^{1/2}\sum_{m_1m_2m_3r}|jm_3\rangle(-1)^{m_1-2m_2+m_3+r} \times\exp(m_3\theta)2^{-j-m_1+r}\left[\frac{(j-m_3)!}{(j+m_3)!}\right]^{1/2} \times\frac{(j+m_1)!(2j-r)!}{(j-m_1)!(m_1-m)!(m_1-m_2)!r!(j+m_2-r)!(j-m_3-r)!}. \tag{A1}$$

Though m_1, m_2, m_3 summations may not be over integers, before performing any summation, we shall ensure that the variable we choose is indeed an integer. Now the m_2 summation can be performed [by first replacing say (m_1-m_2) by a new variable in place of m_2 and summing over it]. This gives

$$E=\exp(-m\theta)\left[\frac{(j-m)!}{(j+m)!}\right]^{1/2}\sum_{m_1m_3r}|jm_3\rangle(-1)^{m_3-m_1+r} \times\exp(m_3\theta)\left[\frac{(j-m_3)!}{(j+m_3)!}\right]^{1/2} \times\frac{(j+m_1)!(2j-r)!}{(j-m_1)!\,(m_1-m)!r!(j+m_1-r)!(j-m_3-r)!}. \tag{A2}$$

The r summation can now be done which results in

$$E=\exp(-m\theta)\left[\frac{(j-m)!}{(j+m)!}\right]^{1/2}\sum_{m_1m_3}|jm_3\rangle(-1)^{m_3-m_1}\exp(m_3\theta) \times\left[\frac{(j+m_3)!}{(j-m_3)!}\right]^{1/2}\frac{1}{(m_1-m)!(-m_1+m_3)!}. \tag{A3}$$

Now the m_1 summation produces δ_{mm_3} which finally gives

$$E=|jm\rangle,$$

exactly the same as on the left-hand side of Eq. (16).

APPENDIX B: EVALUATION OF THE NORMALIZATION COEFFICIENT $\beta(j_1j_2j;\alpha)$

In this Appendix we wish to compute the β's that appeared in Eq. (22) relating the CG coefficients for the quasi-intelligent states with those of the rotation group. As remarked earlier, we should be able to compute β's from the knowledge of the CG coefficient on the left for some special choice of values of the magnetic quantum numbers. From the representations for $|jm\alpha\rangle$ given in Eq. (12) we can, indeed, give an explicit answer for these CG coefficients in terms of the CG coefficients of the rotation group. In general, however, manifesting the factorization expressed in Eq. (22) must be a formidable task. For a very special case, we might, hopefully, be lucky. This is fortunately the case for the special choice $m_1=j_1$, $m_2=-j_2$, $m=j_1-j_2$. Note that since $|j_1-j_2|\leq j\leq j_1+j_2$ the value j_1-j_2 is indeed a permissible value. From the representations in Eq. (12) we obtain

$$|j_1j_1\alpha\rangle=[a^{j_1}{}_{j_1}(\alpha)]^{-1}2^{-j_1}[(2j_1)!]^{1/2}\sum_{m'_1}|j_1m'_1\rangle\frac{\exp(m'_1\theta)}{[(j_1+m'_1)!(j_1-m'_1)!]^{1/2}}, \tag{B1}$$

$$|j_2(-j_2)\alpha\rangle=[a^{j_2}{}_{(-j_2)}(\alpha)]^{-1}2^{-j_2}[(2j_2)!]^{1/2}\sum_{m'_2}|j_2m'_2\rangle(-1)^{j_2+m'_2}\frac{\exp(m'_2\theta)}{[(j_2+m'_2)!(j_2-m'_2)!]^{1/2}}, \tag{B2}$$

and

$$\langle j(j_1-j_2)\alpha|=[a^j{}_{(j_1-j_2)}(\alpha)]^{-1}2^{-j}\left[\frac{(j-j_1+j_2)!}{(j+j_1-j_2)!}\right]^{1/2}\sum_{m'r}|jm'\rangle\left[\frac{(j+m')!}{(j-m')!}\right]^{1/2}(-2)^r\frac{\exp(m'\theta^*)(2j-r)!}{r!(j-j_1+j_2-r)!(j+m'-r)!}. \tag{B3}$$

Thus

$(j(j_1-j_2)\alpha|j_1j_1\alpha;j_2(-j_2)\alpha)$

$$=[a^{j_1}{}_{j_1}(\alpha)a^{j_2}{}_{(-j_2)}(\alpha)a^{j}{}_{(j_1-j_2)}(\alpha)]^{-1}2^{-(j_1+j_2+j)}\left(\frac{(2j_1)!\,(2j_2)!\,(j-j_1+j_2)!}{(j+j_1-j_2)!}\right)^{1/2}\sum_{m_1'm_2'r}\langle jm'|j_1(m'-m_2');j_2m_2'\rangle$$
$$\times\left[\frac{(j+m')!}{(j_1+m'-m_2')!\,(j_1-m'+m_2')!\,(j_2+m_2')!\,(j_2-m_2')!\,(j-m')!}\right]^{1/2}2^r(-1)^{j_2+m_2'+r}\exp[(m'(\theta+\theta^*)]$$
$$\times\frac{(2j-r)!}{r!\,(j-j_1+j_2-r)!\,(j+m'-r)!},\tag{B4}$$

where knowing that the CG coefficient $\langle jm'|j_1m_1';j_2m_2'\rangle$ can be nonzero only when $m'=m_1'+m_2'$, we have eliminated the summation over m_1' by replacing m_1' by $m'-m_2'$ everywhere. Now we use[9]

$$\langle jm'|j_1(m'-m_2');j_2m_2'\rangle$$
$$=\left[\frac{(2j+1)(j-j_1+j_2)!\,(j_1+m'-m_2')!\,(j_1-m'+m_2')!\,(j_2-m_2')!\,(j+m')!}{(j+j_1+j_2+1)!\,(-j+j_1+j_2)!\,(j+j_1-j_2)!\,(j_2+m_2')!\,(j-m')!}\right]^{1/2}$$
$$\times\sum_s(-1)^{-j+j_1+j_2+s}\frac{(j_2+m_2'+s)!\,(j+j_1-m_2'-s)!}{s!\,(j_1+m'-m_2'-s)!\,(j_2-m_2'-s)!\,(j-j_1+m_2'+s)!}\tag{B5}$$

for the CG coefficient of the rotation group which appears above in Eq. (B4). This results is

$$(j(j_1-j_2)\alpha|j_1j_1\alpha;j_2(-j_2)\alpha)$$
$$=[a^{j_1}{}_{j_1}(\alpha)a^{j_2}{}_{(-j_2)}(\alpha)a^{j}{}_{(j_1-j_2)}(\alpha)]^{-1}\left[\frac{(2j+1)(2j_1)!\,(2j_2)!}{(j+j_1+j_2+1)!\,(j_1+j_2-j)!}\right]^{1/2}\left\{\frac{(j-j_1+j_2)!}{(j+j_1-j_2)}\sum_{m'm_2'rs}(-1)^{-j+j_1+2j_2+m_2'+r+s}2^{-(j+j_1+j_2)+r}\right.$$
$$\left.\times\frac{(j+m')!}{(j-m')!\,(j_2+m_2')!}\exp[m'(\theta+\theta^*)]\frac{(2j-r)!\,(j_2+m_2'+s)!\,(j+j_1-m_2'-s)!}{r!\,(j-j_1+j_2-r)!\,(j+m'-r)!\,s!\,(j_1+m'-m_2'-s)!\,(j_2-m_2'-s)!\,(j-j_1+m_2'+s)!}\right\}.\tag{B6}$$

The expression within the curly brackets is called S in the following. To simplify S, we replace $j_2-m_2'-s$ by s, i.e., we replace s by $j_2-m_2'-s$. This gives

$$S=\frac{(j-j_1+j_2)!}{(j+j_1-j_2)!}\sum_{m'm_2'rs}2^{-(j+j_1+j_2)+r}(-1)^{-j+j_1-j_2+r+s}\frac{(j+m')!}{(j_2+m_2')!\,(j-m')!}\exp[m'(\theta+\theta^*)]$$
$$\times\frac{(2j-r)!\,(2j_2-s)!\,(j+j_1-j_2+s)!}{r!\,(j-j_1+j_2-r)!\,(j+m'-r)!\,s!\,(j_1-j_2+m'+s)!\,(j_2-m_2'-s)!\,(j-j_1+j_2-s)!}.\tag{B7}$$

The m_2' summation now gives $2^{2j_2-s}/(2j_2-s)!$ This results in

$$S=\frac{(j-j_1+j_2)!}{(j+j_1-j_2)!}\sum_{m'rs}2^{-(j+j_1-j_2)+r-s}(-1)^{-j+j_1-j_2+r+s}\frac{(j+m')!}{(j-m')!}\exp[m'(\theta+\theta^*)]$$
$$\times\frac{(2j-r)!\,(j+j_1-j_2+s)!}{r!\,(j-j_1+j_2-r)!\,(j+m'-r)!\,s!\,(j_1-j_2+m'+s)!\,(j-j_1+j_2-s)!}.\tag{B8}$$

To put S in a form which can be recognized, we replace $j-j_1+j_2-s$ by a new variable s which results in

$$S=\frac{(j-j_1+j_2)!}{(j+j_1-j_2)!}\sum_{m'rs}\frac{(j+m')!}{(j-m')!}\exp[m'(\theta+\theta^*)]2^{-2j+r+s}(-1)^{r+s}$$
$$\times\frac{(2j-r)!\,(2j-s)!}{r!\,(j-j_1+j_2-r)!\,(j+m'-r)!\,s!\,(j-j_1+j_2-s)!\,(j+m'-s)!}.\tag{B9}$$

Comparing the above form of S with Eq. (12a), we immediately conclude

$$S=[a^{j}{}_{(j_1-j_2)}(\alpha)]^2.$$

Now we return to Eq. (B6). Recognizing that

$$\langle j(j_1-j_2)|j_1j_1;j_2(-j_2)\rangle=\left(\frac{(2j+1)(2j_1)!\,(2j_2)!}{(j+j_1+j_2+1)!\,(j_1+j_2-j)!}\right)^{1/2},\tag{B10}$$

we find

$$(j(j_1-j_2)\alpha|j_1j_1\alpha;j_2(-j_2)\alpha)=\frac{a^{j}{}_{(j_1-j_2)}(\alpha)}{a^{j_1}{}_{j_1}(\alpha)a^{j_2}{}_{(-j_2)}(\alpha)}\langle j(j_1-j_2)|j_1j_1;j_2(-j_2)\rangle,\tag{B11}$$

which on comparison with Eq. (22) shows that $\beta(j_1j_2j;\alpha)=1$. Note that it has been fixed completely by the *phase convention* used in *defining* the relationship [Eqs. (12)] between the quasi-intelligent states and the Wigner states. This equation has a built in *phase convention* which *cannot* be fixed by knowing *only* that $|jm\alpha\rangle$ is an eigenstate of the operator $J_3'(\alpha)$.

In the above computations, we have used the fact that the Clebsch—Gordan coefficient $(j(j_1-j_2)\alpha|j_1j_1\alpha;j_2(-j_2)\alpha)$

is just the inner product $\langle j(j_1-j_2)\alpha\,|\,(|j_1j_2\alpha\rangle\,|j_2(-j_2)\alpha\rangle)\rangle$. We could have also used that this coefficient is also equal to the inner product $\langle j(j_1-j_2)\alpha^c\,|\,(|j_1j_1\alpha\rangle\,|j_2(-j_2)\alpha\rangle)\rangle[a^j{}_{j_1-j_2}(\alpha)]^2$. It is obvious from (B9) above that the S corresponding to this new inner product would have been 1 and we would be able to reproduce the previous results.

APPENDIX C: AN ANALOG OF THE EXPANSION OF THE UNIT OPERATOR

In this Appendix, we shall prove that

$$\sum_m |jm\alpha\rangle\langle jm\alpha|[a^j{}_m(\alpha)]^2=\sum_m |jm\rangle\langle jm|\exp[m(\theta+\theta^*)]. \tag{C1}$$

Using Eq. (12a) for $|jm\alpha\rangle$ and Eq. (12b) for $\langle jm\alpha|$ we get

$$\sum_m |jm\alpha\rangle\langle jm\alpha|[a^j{}_m(\alpha)]^2=\sum_{mm_1m_2rs}|jm_1\rangle\langle jm_2|\exp(m_1\theta+m_2\theta^*)\left[\frac{(j+m_1)!\,(j-m_2)!}{(j-m_1)!\,(j+m_2)!}\right]^{1/2}2^{-2j+r+s}$$
$$\times(-1)^{m_2-m+r+s}\frac{(2j-r)!\,(2j-s)!}{r!\,(j-m-r)!\,(j+m_1-r)!\,s!\,(j+m-s)!\,(j-m_2-s)!}. \tag{C2}$$

Now

$$\sum_m (-1)^{j-m-r}\frac{1}{(j-m-r)!\,(j+m-s)!}=\delta_{2j-r-s,0}. \tag{C3}$$

From the above two equations, we obtain

$$\sum_m |jm\alpha\rangle\langle jm\alpha|[a^j{}_m(\alpha)]^2=\sum_{m_1m_2r}|jm_1\rangle\langle jm_2|\exp(m_1\theta+m_2\theta^*)\left[\frac{(j+m_1)!\,(j-m_2)!}{(j-m_1)!\,(j+m_2)!}\right]^{1/2}(-1)^{j+m_2-r}$$
$$\times\frac{1}{(j+m_1-r)!\,(-j-m_2+r)!}. \tag{C4}$$

On performing the trivial r summation, we obtain a delta function $\delta_{m_1m_2}$ and finally arrive at Eq. (C1) which is a *generalization* of the expansion of a unit operator in the sense that if we restricted ourselves to real $\theta=0$, we would obtain as a *special* case of Eq. (C1)

$$\sum_m |jm\alpha\rangle\langle jm\alpha|=\sum |jm\rangle\langle jm|=I, \tag{C5}$$

which is indeed an expansion of the unit operator. We did use the fact that for real $\theta=0$, $a^j{}_m(\alpha)=1$. But in this case, the states $|jm\alpha\rangle$ are indeed *orthonormalized* by just writing

$$|jm\alpha\rangle=\exp(\theta J_3)\exp(-i\tfrac{1}{2}\pi J_2)|jm\rangle$$

and Eq. (C5) should be obvious.

Incidentally, a proof similar to the above results in

$$\sum_m |jm\alpha^c\rangle\langle jm\alpha|[a^j{}_m(\alpha)]^2=\sum_m |jm\rangle\langle jm|=I. \tag{C6}$$

[1]M.A. Rashid, J. Math. Phys. **19**, 1391 (1978).
[2]This restriction is explained in Ref. 1.
[3]See Eq. (29) in Ref. 1.
[4]The normalization of the operator $J_3'(\alpha)$ has been chosen to have the same spectrum as that of J_3.
[5]Note that our normalization of the operators makes the answers of their effects on the quasi-intelligent states as very simple.
[6]Note that $J_3'(\alpha^c)=[J_3'(\alpha)]^\dagger$, whereas $J_\pm'(\alpha^c)=[J_\mp'(\alpha)]^\dagger$.
[7]See Eqs. (32b) and (32c) in Ref. 1.
[8]We shall make this choice by invoking consistency with Eqs. (12).
[9]$\langle jm'\,|\,j_1(m'-m_2');j_2m_2'\rangle$
$=(-1)^{j_1+j_2-j}(-1)^{j_2+m_2'}\left(\frac{2j+1}{2j_1+1}\right)\langle j_2m_2';j(-m')\,|\,j_1(-m'+m_2')\rangle$
on combining Eqs. (3.5.15) and (3.5.17) in A.R. Edmonds, in *Angular Momentum in Quantum Mechanics* (Princeton U.P., Princeton, New Jersey, 1957). Finally we used Eq. (3.6.10) for the CG coefficient on the right. This premanipulation of the CG coefficient has reduced the size of the Appendix considerably.