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Minimum Uncertainty Product, Number–Phase Uncertainty Product, and Coherent States

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The number–phase uncertainty products proposed by Carruthers and Nieto are studied to determine whether they are minimized by coherent states. It is found that coherent states do not minimize these products. States that do minimize some of the uncertainty products are constructed. Variational techniques for the study of arbitrary uncertainty products are developed.

I. INTRODUCTION

Recent discussions of a quantum-mechanical phase operator for harmonic oscillators have shown that a Hermitian phase operator φ_{op} does not exist.[1] Susskind and Glogower[1] (SG) have demonstrated however that Hermitian sine (S) and cosine (C) operators can be defined which have many properties that are suggested by the nomenclature. Carruthers and Nieto[2] (CN) have examined the matrix elements of S and C between Glauber's[3] coherent states. They found that in the high-excitation (classical) limit the expectation values of S and C, in these states, behave as the sine and cosine of the phase of the harmonic oscillation.

Carruthers and Nieto have also proposed several uncertainty relations to replace the traditional expression for the number–phase uncertainty product

$$(\Delta N)^2(\Delta\varphi)^2 \geq \tfrac{1}{4}, \tag{1}$$

which is ill-defined. The proposed uncertainty products are given in terms of the S and C operators and have the virtue that, when evaluated with coherent states, they approach their theoretic minimum for highly excited coherent states, and remain small for moderate excitation.

In this paper, we examine further the uncertainty products given by CN, in order to determine whether the coherent states do in fact give the smallest uncertainty product. Towards this end, we develop in Sec. II new variational techniques for determining those normalizable states which give a minimum for the uncertainty product of noncommuting Hermitian operators. In Sec. III we show that the coherent states do not minimize the number–phase uncertainty products. We also construct states which do have the desired property. In Sec. IV we examine the S–C uncertainty product.

II. MINIMUM UNCERTAINTY PRODUCT

A. When two Hermitian operators X and Y do not commute, they cannot be simultaneously diagonalized and their uncertainty product satisfies the inequality

$$(\Delta X)^2(\Delta Y)^2 \geq \tfrac{1}{4}\langle A\rangle^2. \tag{2}$$

Here $(\Delta X)^2 \equiv \langle X^2\rangle - \langle X\rangle^2$, and iA is the commutator of X and Y, assumed to be nonzero. A procedure for determining the state for which the uncertainty product, appearing in (2), is minimized was given by

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[1] L. Susskind and J. Glogower, Physics 1, 49 (1964).
[2] P. Carruthers and M. M. Nieto, Phys. Rev. Letters 14, 387 (1965).
[3] R. J. Glauber, Phys. Rev. 131, 2766 (1963).

Heisenberg when X and Y are position and momentum operators. His method holds generally when A is a c number, and is described in any textbook on quantum mechanics. However, if A is not a c number, Heisenberg's method cannot, in general, be used to obtain the minimizing state. We briefly summarize here Heisenberg's method, show that it is inapplicable when A is a q number, and then develop a variational method appropriate to the case when A is a q number. This latter method is also applicable when the uncertainty product no longer has the simple form (2).

B. Heisenberg's method consists of establishing that for any normalizable state $|\Psi\rangle$,

$$U(\Psi) \equiv (\Delta X)^2(\Delta Y)^2 = \langle \hat{X}^2\rangle\langle \hat{Y}^2\rangle = |\langle \hat{X}\hat{Y}\rangle|^2 + R(\Psi) = \tfrac{1}{4}P(\Psi) + \tfrac{1}{4}Q(\Psi) + R(\Psi), \quad (3a)$$

where

$$\hat{X} \equiv X - \langle X\rangle,$$
$$P(\Psi) \equiv |\langle[\hat{X}, \hat{Y}]\rangle|^2 = |\langle[X, Y]\rangle|^2 = \langle A\rangle^2,$$
$$Q(\Psi) \equiv |\langle\{\hat{X}, \hat{Y}\}\rangle|^2. \quad (3b)$$

The term $R(\Psi)$ in (3a) is a positive semidefinite remainder term, arising from an application of the Schwartz inequality to $\langle \hat{X}^2\rangle\langle \hat{Y}^2\rangle$. $R(\Psi)$ vanishes if and only if $\hat{X}|\Psi\rangle$ is proportional to $\hat{Y}|\Psi\rangle$. $P(\Psi)$ and $Q(\Psi)$ are also positive semidefinite, depending in general on Ψ. However, if A is a c number, $P(\Psi)$ does not depend on Ψ since $\langle\Psi \mid \Psi\rangle = 1$. In this case

$$U(\Psi) = \tfrac{1}{4}A^2 + \tfrac{1}{4}Q(\Psi) + R(\Psi). \quad (4)$$

Clearly the minimum value for U is $\frac{1}{4}A^2$, which is reached if and only if Q and R vanish. Combining the requirement that R vanishes with the requirement that Q vanishes gives an equation for $|\Psi\rangle$:

$$\hat{X}|\Psi\rangle + i\gamma\hat{Y}|\Psi\rangle = 0, \quad \gamma \text{ real}, \quad (5a)$$

or

$$[X + i\gamma Y]|\Psi\rangle = \lambda|\Psi\rangle, \quad \lambda \equiv \langle X\rangle + i\gamma\langle Y\rangle. \quad (5b)$$

For future reference, we give another equation for $|\Psi\rangle$ which follows from (5a) upon multiplication by $\hat{X} - i\gamma\hat{Y}$:

$$[\hat{X}^2 + \gamma^2\hat{Y}^2 - \gamma A]|\Psi\rangle = 0. \quad (6a)$$

Equation (6a) shows that γ may be evaluated in terms of matrix elements of X and Y. By premultiplying (6a) by $\langle\Psi|$, we get

$$\gamma = \frac{A}{2(\Delta Y)^2} = \pm\left(\frac{(\Delta X)^2}{(\Delta Y)^2}\right)^{\frac{1}{2}}. \quad (6b)$$

In obtaining (6b), we have used the fact that A is a c number and $(\Delta X)^2(\Delta Y)^2 = \frac{1}{4}A^2$. (We assume always that $(\Delta X^2)(\Delta Y^2) \neq 0$; viz., $|\Psi\rangle$ is not an eigenstate of X or Y.[4])

Equation (5b) is to be solved as an eigenvalue equation for $|\Psi\rangle$, with three free parameters: γ, Re λ, Im λ, and a subsidiary normalization condition $\langle\Psi \mid \Psi\rangle = 1$. The state $|\Psi\rangle$ then minimizes the uncertainty product (3c). We refer to this method as the *direct method* for obtaining the minimizing state.

C. The direct method is not in general applicable when A is a q number. In that eventuality, $P(\Psi)$ *does* depend on Ψ, and we cannot conclude that a minimum for U is achieved when Q and R vanish.

We now solve the minimization problem without assuming that the commutator of X and Y is a c number. For an uncertainty product of the form $(\Delta X)^2(\Delta Y)^2$, the problem is nontrivial only when the matrix elements of X between eigenstates of Y diverge, and vice versa. For if they are finite, the uncertainty product is manifestly minimized to zero when it is evaluated with eigenstates of X or Y. The subsequent analysis applies only to the nontrivial problem. However, later we generalize it to the case when the uncertainty products are not of the simple form $(\Delta X)^2(\Delta Y)^2$ and it no longer is obvious how to find the minimizing state, even though all matrix elements are finite.

Since the expression for $U(\Psi)$ given in (3c) involves the function $R(\Psi)$ about which we have no useful information, we return to the definition of $U(\Psi)$ in terms of matrix elements of X and Y. If $U(\Psi)$ is to be minimized, we may apply the variation principle and require that $U(\Psi)$ be stationary under arbitrary variations of Ψ. With the help of a Lagrange multiplier m, we also impose the subsidiary condition that $\langle\Psi \mid \Psi\rangle = 1$. Considering the variation of $|\Psi\rangle$ to be independent of $\langle\Psi|$, we obtain as a necessary condition for $U(\Psi)$ to be a minimum:

$$\delta U/\delta\langle\Psi| = m|\Psi\rangle. \quad (7a)$$

Since $U \equiv (\Delta X)^2(\Delta Y)^2$, we need to evaluate $\delta(\Delta X)^2/\delta\langle\Psi|$. From the definition of $(\Delta X)^2$, we have

$$(\Delta X)^2 = \langle\Psi|X^2|\Psi\rangle - \langle\Psi|X|\Psi\rangle^2,$$
$$\delta(\Delta X)^2/\delta\langle\Psi| = X^2|\Psi\rangle - 2X|\Psi\rangle\langle X\rangle = [X - \langle X\rangle]^2|\Psi\rangle - \langle X\rangle^2|\Psi\rangle. \quad (7b)$$

Therefore (7a) becomes

$$(\Delta Y)^2\hat{X}^2|\Psi\rangle + (\Delta X)^2\hat{Y}^2|\Psi\rangle = ((\Delta Y)^2\langle X\rangle^2 + (\Delta X)^2\langle Y\rangle^2 + m)|\Psi\rangle. \quad (7c)$$

[4] If $|\Psi\rangle$ is an eigenstate of one of the two operators, say X, then $(\Delta X)^2 = 0$, and $(\Delta Y)^2$ necessarily diverges when A is a c number. Therefore the uncertainty product is of the indeterminate form $0 \cdot \infty$. We find in our subsequent analysis that such an indeterminate quantity can be sometimes evaluated; see Appendix A.

Finally, by taking matrix elements of the above with $\langle\Psi|$, we evaluate m, and discover that the coefficient on the right-hand side is $2(\Delta X)^2(\Delta Y)^2$, which we assume to be nonzero. Thus we obtain an Euler–Lagrange (EL)-type equation for $|\Psi\rangle$ which must be satisfied if U is to be a minimum:

$$\left[\frac{\hat{X}^2}{(\Delta X)^2}+\frac{\hat{Y}^2}{(\Delta Y)^2}-2\right]|\Psi\rangle=0. \tag{8}$$

We call the states $|\Psi\rangle$ which solve (8) *critical states.*[5] Evidently every critical state $|\Psi\rangle$ makes $U(\Psi)$ stationary.

This simple equation provides the appropriate generalization of the direct method to the case that $[X, Y]$ is a q number. For future reference we call this the *analytic method.* Equation (8) is to be solved as an eigenvalue equation with four free, real parameters, viz.,

$$\left[\frac{(X-\alpha)^2}{a^2}+\frac{(Y-\beta)^2}{b^2}-2\right]|\Psi\rangle=0. \tag{9a}$$

Once $|\Psi\rangle$ has been obtained from (9a), the four parameters are determined self-consistently by setting[6]

$$\alpha=\langle X\rangle, \qquad \beta=\langle Y\rangle,$$
$$a^2=\langle X^2\rangle-\langle X\rangle^2, \quad b^2=\langle Y^2\rangle-\langle Y\rangle^2. \tag{9b}$$

In general, since Eq. (9a) is an eigenvalue equation with self-consistency conditions (9b), we expect to obtain solutions only when a special (eigenvalue) relation exists between the parameters. Nevertheless, we may expect to obtain more than one solution, since Eq. (8) serves equally well to determine other stationary points of U: further minima, maxima, or "points of inflection" of U. One must therefore examine $(\Delta X)^2(\Delta Y)^2=a^2b^2$ for the various critical states, to determine which gives the smallest value. (If it is not evident that a minimum has indeed been attained, one might compute the second variation of U to determine the nature of the stationary point.)

Having established a necessary condition on $|\Psi\rangle$ for $U(\Psi)$ to be a minimum, we may examine the direct method critically to establish its precise relation to the analytic method. Suppose we set out to determine a state $|\Psi\rangle$ by the direct method (regardless of the nature of A). Then (6a) is valid, which may now be written as

$$\left[\frac{\hat{X}^2}{(\Delta X)^2}+\frac{\gamma^2Y^2}{(\Delta X)^2}\hat{Y}^2-\frac{\gamma A}{(\Delta X)^2}\right]|\Psi\rangle=0. \tag{10a}$$

The parameter γ may be again evaluated by taking matrix elements and remembering that (within the direct method) $(\Delta X)^2(\Delta Y)^2=\frac{1}{4}\langle A\rangle^2$. Then [compare (6b)],

$$\gamma=\frac{\langle A\rangle}{2(\Delta Y)^2}=\pm\left(\frac{(\Delta X)^2}{(\Delta Y)^2}\right)^{\frac{1}{2}} \tag{10b}$$

and (10a) becomes

$$\left[\frac{\hat{X}^2}{(\Delta X)^2}+\frac{\hat{Y}^2}{(\Delta Y)^2}-\frac{2A}{\langle A\rangle}\right]|\Psi\rangle=0. \tag{10c}$$

Comparing this to (8), we see that the direct method determines a critical state $|\Psi\rangle$ which corresponds to a stationary value of $U(\Psi)$, if and only if $|\Psi\rangle$ is an eigenstate of A.

In conclusion we note that even when A is a c number and the direct method is applicable, Eq. (5) may not have a solution. Then U never achieves its minimum of $\frac{1}{4}A^2$. Nevertheless, it may achieve some lowest value which is greater than $\frac{1}{4}A^2$; and the analytic method may be used to determine the states for which this occurs.

D. In order to exhibit the workings of our analytic method, we solve the classic problem of minimizing the position–momentum uncertainty product

$$(\Delta x)^2(\Delta p)^2.$$

In obtaining this old result, we find all the critical states for which $(\Delta x)^2(\Delta p)^2$ is stationary.

According to (8) we must solve ($\hbar=1$)

$$\left[\frac{(x-\alpha)^2}{a^2}+\frac{[(1/i)\partial/\partial x-\beta]^2}{b^2}\right]\Psi(x)=2\Psi(x). \tag{11a}$$

The solution to (11a) can be found by comparison to the Schrödinger equation for a harmonic oscillator. Therefore (11a) possesses normalized solutions only when

$$|ab|=\tfrac{1}{2}(2n+1). \tag{11b}$$

The normalized solutions are

$$\Psi_n(x)=e^{i\beta x}U_n(x-\alpha), \tag{11c}$$

where $U_n(x)$ is a normalized harmonic-oscillator eigenfunction, with mass $\frac{1}{2}b^2$, stiffness constant $2/a^2$ and energy 2. The self-consistency requirements (9b) set no further conditions beyond (11b), and all the Ψ_n's are critical states for $(\Delta x)^2(\Delta p)^2$. Evidently the

[5] Eigenstates of X and Y pose a special problem. For suppose we take $|\Psi\rangle$ to be an eigenstate of X, and assume that $(\Delta Y)^2$ diverges so that the problem is nontrivial. Then Eq. (8) has the indeterminate form $0/0\,|\Psi\rangle+\hat{Y}^2/\infty\,|\Psi\rangle-2|\Psi\rangle=0$. Evidently an effective point of view is to ignore those solutions of (8) which are eigenstates of X and Y, and evaluate U separately with the eigenstates to determine whether these minimize U.

[6] In the direct method, the parameters λ and γ need not be evaluated separately since their value is set by the form of Eq. (5a). Indeed the four conditions in (9b) are redundant since the form of Eq. (9a) assures that one relation between the parameters exists, viz.,

$$\frac{\langle X^2\rangle-2\alpha\langle X\rangle+\alpha^2}{a^2}+\frac{\langle Y^2\rangle-2\beta\langle Y\rangle+\beta^2}{b^2}=2.$$

minimum uncertainty product is $\frac{1}{4}$, which is attained with the state Ψ_0:

$$\Psi_0(x) = [2\pi(\Delta x)^2]^{-\frac{1}{4}} \exp\{-[(x - \langle x\rangle)^2/(\Delta x)^2]\}. \quad (11d)$$

The fact that an oscillator ground-state wavefunction minimizes the position–momentum uncertainty product is well known, and has been considered to be a fortuitous coincidence. It is seen from the present analysis that this result is a natural consequence of our analytic method. Moreover, we have obtained the further knowledge that all the harmonic-oscillator wavefunctions are critical states which make $(\Delta x)^2(\Delta p)^2$ stationary.

E. We continue with our discussion of the minimization problem for uncertainty products by discussing objects which are of a form more complicated than $(\Delta X)^2(\Delta Y)^2$. (Such uncertainty products have been proposed by CN.)

If the commutator of X and Y is not a c number, it may be of consequence to consider an uncertainty product of the form

$$U_1(\Psi) \equiv \frac{(\Delta X)^2(\Delta Y)^2}{\langle A\rangle^2} = \frac{U(\Psi)}{\langle A\rangle^2}. \quad (12a)$$

By applying the variation principle, we immediately obtain the necessary condition on $|\Psi\rangle$, for which $U_\perp(\Psi)$ is stationary:

$$\left[\frac{\hat{X}^2}{(\Delta X)^2} + \frac{\hat{Y}^2}{(\Delta Y)^2} - \frac{2A}{\langle A\rangle}\right]|\Psi\rangle = 0. \quad (12b)$$

This equation is the same as Eq. (10b) which follows from the direct method. Indeed, that the direct method is applicable, may be seen by reference to Eq. 3c).

According to that expression,

$$U_1(\Psi) = \frac{1}{4} + \frac{1}{4}\frac{Q(\Psi)}{P(\Psi)} + \frac{R(\Psi)}{P(\Psi)}. \quad (12c)$$

Thus when we arrange for Q and R to vanish, as is done in the direct method, U_1 attains its minimum. [We must of course examine separately the situation if the direct method yields a solution for which $P(\Psi) = 0$.]

When the expression for the uncertainty product is even more complicated, for example if it involves the matrix elements of more than two operators, the direct method, even if applicable, will not in general yield solutions. The variation principle may nevertheless be used to give a (complicated) necessary condition.

III. NUMBER-PHASE UNCERTAINTY PRODUCTS

A. We now turn to the number–phase uncertainty products proposed by CN. Following SG,[1] we consider a harmonic oscillator described by creation and annihilation operators a and $a^\dagger$, respectively, which obey $[a, a^\dagger] = 1$. The number operator $N_{\text{op}} \equiv a^\dagger a$ has number states $|n\rangle$ as eigenvectors; and $a\,|n\rangle = n^{\frac{1}{2}}\,|n-1\rangle$, $a^\dagger\,|n\rangle = (n+1)^{\frac{1}{2}}\,|n+1\rangle$, $a^\dagger a\,|n\rangle = n\,|n\rangle$. The eigenvectors of a are the coherent states $|\alpha\rangle$: $a\,|\alpha\rangle = \alpha\,|\alpha\rangle$. They have the property that $N \equiv \langle\alpha|\,N_{\text{op}}\,|\alpha\rangle = |\alpha|^2 = \langle\alpha|\,N_{\text{op}}^2\,|\alpha\rangle - \langle\alpha|\,N_{\text{op}}\,|\alpha\rangle^2 \equiv (\Delta N)^2$. In terms of number states, the coherent states are given by

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{\frac{1}{2}}}\,|n\rangle. \quad (14)$$

Evidently each coherent state may be described by two parameters: amplitude and phase of α. Thus we frequently write $|N\varphi\rangle$ for $|\alpha\rangle$ where $\alpha = N^{\frac{1}{2}}e^{i\varphi}$. To define the sine and cosine operators, we define first the number state raising and lowering operators $E_\pm$:

$$E_- \equiv (N_{\text{op}} + 1)^{-\frac{1}{2}},$$
$$E_+ \equiv a^\dagger(N_{\text{op}} + 1)^{-\frac{1}{2}} = (E_-)^\dagger. \quad (15a)$$

These satisfy

$$\begin{aligned} E_\pm\,|n\rangle &= |n \pm 1\rangle\ n \neq 0,\\ E_+\,|0\rangle &= |1\rangle, \quad E_-\,|0\rangle = 0,\\ E_-E_+ &= I, \qquad E_+E_- = I - P,\\ [E_-, E_+] &= P, \qquad P\,|n\rangle = \delta_{n0}\,|0\rangle. \end{aligned} \quad (15b)$$

The S and C operators then are defined by

$$C = \tfrac{1}{2}[E_- + E_+], \quad S = \tfrac{1}{2}i^{-1}[E_- - E_+], \quad (16a)$$

$$[N_{\text{op}}, S] = iC, \quad [N_{\text{op}}, C] = -iS, \quad [S, C] = \frac{1}{2i}P. \quad (16b)$$

For coherent states, the matrix elements of C, S, C^2, S^2 are given by

$$I_c \equiv \langle N\varphi|\,C\,|N\varphi\rangle = I(N)\cos\varphi,$$
$$I_s \equiv \langle N\varphi|\,S\,|N\varphi\rangle = I(N)\sin\varphi,$$
$$I(N) \equiv N^{\frac{1}{2}}e^{-N}\sum_n\frac{N^n}{n!\,(n+1)^{\frac{1}{2}}}, \quad (17a)$$

$$\begin{aligned} J_c &\equiv \langle N\varphi|\,C^2\,|N\varphi\rangle\\ &= \tfrac{1}{2} - \tfrac{1}{4}e^{-N} + \tfrac{1}{2}(\cos^2\varphi - \sin^2\varphi)J(N),\\ J_s &\equiv \langle N\varphi|\,S^2\,|N\varphi\rangle\\ &= \tfrac{1}{2} - \tfrac{1}{4}e^{-N} - \tfrac{1}{2}(\cos^2\varphi - \sin^2\varphi)J(N), \end{aligned}$$

$$J(N) \equiv Ne^{-N}\sum_{n=0}^{\infty}\frac{N^n}{n!\,((n+1)(n+2))^{\frac{1}{2}}}. \quad (17b)$$

The functions $I(N)$ and $J(N)$ have the asymptotic (large N) forms

$$I(N) \approx 1 - \frac{1}{8N}, \quad J(N) \approx 1 - \frac{1}{2N}. \quad (17c)$$

Hence for large N,

$$I_c \approx \cos\varphi, \quad I_s \approx \sin\varphi,$$
$$J_c \approx \cos^2\varphi, \quad J_s \approx \sin^2\varphi. \tag{17d}$$

It is seen that the expressions involving S may be obtained from those involving C by replacing φ by $\frac{1}{2}\pi - \varphi$.

The uncertainty relations proposed by CN are

$$\begin{aligned} U_1(\Psi) &\equiv (\Delta N)^2(\Delta C)^2/\langle S\rangle^2 \geq \tfrac{1}{4}, \\ U_2(\Psi) &\equiv (\Delta N)^2(\Delta S)^2/\langle C\rangle^2 \geq \tfrac{1}{4}, \\ U_3(\Psi) &\equiv (\Delta N)^2 \frac{(\Delta S)^2 + (\Delta C)^2}{[\langle S\rangle^2 + \langle C\rangle^2]} \geq \tfrac{1}{4}. \end{aligned} \tag{18}$$

These relations have the virtues that (i) they represent plausible generalizations of the imprecise statement $(\Delta N)^2(\Delta\varphi)^2 \geq \frac{1}{4}$; (ii) for highly excited coherent states they closely approach their theoretical lower limit $\frac{1}{4}$ and remain small for moderate excitations. The last uncertainty product is independent of φ when evaluated with coherent states.

B. It is demonstrably true that the coherent states do not permit the U_i's to attain their theoretical lower limit $\frac{1}{4}$. It may nevertheless be the case that no normalizable states exist for which $U_i = \frac{1}{4}$; and the coherent states give the lowest attainable minimum. We establish that (i) the coherent states are not critical states, viz., they do not make the uncertainty products stationary; therefore *a fortiori* they do not minimize the uncertainty products; and (ii) normalizable states exist which allow some of the U_i's to reach their theoretical lower limit of $\frac{1}{4}$.

C. We first study U_1. The critical states, which make U_1 stationary, satisfy according to (12b)

$$0 = \left[\frac{[N_{\text{op}} - \langle N\rangle]^2}{(\Delta N)^2} + \frac{[C - \langle C\rangle]^2}{(\Delta C)^2} - \frac{2S}{\langle S\rangle}\right]|\Psi\rangle. \tag{19}$$

Expanding $|\Psi\rangle$ in number states $|\Psi\rangle = \sum_n a_n |n\rangle$, we find that the coefficients a_n must satisfy the recursion relation

$$\begin{aligned} &\tfrac{1}{4}a_2 + \left(\frac{ib^2}{\gamma} - \beta\right)a_1 + \left(\frac{b^2}{a^2}\alpha^2 + \tfrac{1}{4} + \beta^2\right)a_0 = 0, \\ &\tfrac{1}{4}a_{n+2} + \left(\frac{ib^2}{\gamma} - \beta\right)a_{n+1} + \left(\frac{b^2}{a^2}(n-\alpha)^2 + \tfrac{1}{2} + \beta^2\right)a_n \\ &\quad - \left(\frac{ib^2}{\gamma} + \beta\right)a_{n-1} + \tfrac{1}{4}a_{n-2} = 0 \quad n \geq 1, \quad a_{-1} = 0, \end{aligned} \tag{20a}$$

subject to the subsidiary conditions

$$\begin{aligned} 1 &= \langle\Psi|\Psi\rangle, \quad \alpha \equiv \langle\Psi| N_{\text{op}}|\Psi\rangle, \\ \beta &\equiv \langle\Psi| C|\Psi\rangle, \quad \gamma \equiv \langle\Psi| S|\Psi\rangle, \\ a^2 &\equiv (\Delta N)^2 = \langle\Psi| N_{\text{op}}^2|\Psi\rangle - \alpha^2, \\ b^2 &\equiv (\Delta C)^2 = \langle\Psi| C^2|\Psi\rangle - \beta^2. \end{aligned} \tag{20b}$$

Whether the coherent states satisfy these equations can be easily checked by setting $|\Psi\rangle = |N\varphi\rangle$; viz.,

$$a_n = e^{-\frac{1}{2}N}[N^{\frac{1}{2}n}e^{in\varphi}/(n!)^{\frac{1}{2}}], \quad \alpha = a^2 = N.$$

For simplicity we also assume $\varphi = \frac{1}{2}\pi$; viz., $\beta = 0$. Then (20a) becomes

$$\frac{N}{\sqrt{2}} + \frac{4b^2}{\gamma}N^{\frac{1}{2}} - (4b^2N + 1) = 0, \tag{21a}$$

$$\begin{aligned} &\frac{N^2}{((n+2)(n+1)n(n-1))^{\frac{1}{2}}} + 4\frac{b^2N^{\frac{3}{2}}}{\gamma((n+1)n(n-1))^{\frac{1}{2}}} \\ &\quad - \left(\frac{4b^2}{N}[n-N]^2 + 2\right)\frac{N}{(n(n+1))^{\frac{1}{2}}} \\ &\quad + \frac{4b^2}{\gamma}\frac{N^{\frac{1}{2}}}{(n)^{\frac{1}{2}}} + 1 = 0. \end{aligned} \tag{21b}$$

Equations (21) are manifestly not satisfied; hence the coherent states are not critical states, and do not minimize the uncertainty product.

We may however demonstrate that for large N the coherent states do satisfy (21b) approximately. For large N, $(\Delta N)^2(\Delta C)^2 \sim \frac{1}{4}\langle S\rangle^2 \sim \frac{1}{4}\sin^2\varphi = \frac{1}{4}$; thus $4b^2 = 4(\Delta C)^2 \sim \langle S\rangle^2/\langle\Delta N\rangle^2 \sim 1/N$; and $\gamma \sim \sin\varphi = 1$ [see (17)]. Also for large N the most important number states $|n\rangle$, contributing to the coherent state $|N\varphi\rangle$, are those with $n \sim N$; since for these values of n, $N^{\frac{1}{2}}(n!)^{-\frac{1}{2}}$ assumes its maximum. Therefore for large N and $n \sim N$ the left-hand side of (21b) becomes $O(1/N)$, and (21b) is approximately satisfied. This argument cannot be given when $\langle C\rangle \equiv \beta \neq 0$.

It is evident that the analysis of U_2 proceeds in the same fashion towards the same conclusion, except that the condition $\langle C\rangle = 0$ is now replaced by $\langle S\rangle = 0$.

D. States that do minimize the uncertainty product U_1 and allow it to achieve its theoretical lower limit of $\frac{1}{4}$ may be easily constructed (under certain restrictions). The discussion in Sec. IIE shows that we may use the direct method to determine these states. Accordingly we wish to solve

$$\begin{aligned} (N_{\text{op}} + i\gamma C)|\Psi\rangle &= \lambda|\Psi\rangle, \\ \langle\Psi|\Psi\rangle &= 1. \end{aligned} \tag{22}$$

For simplicity, we again confine ourselves to the case $\langle\Psi| C|\Psi\rangle = 0$. This makes λ real and equal to $\langle\Psi| N_{\text{op}}|\Psi\rangle$. Expanding in number states leads to the recursion relation

$$\begin{aligned} (\lambda - n)a_n &= \tfrac{1}{2}i\gamma(a_{n+1} + a_{n-1}), \\ a_{-1} &= 0. \end{aligned} \tag{23a}$$

To put this in a more transparent form, we define

$a_n = (-i)^n b_n$; then the b_n's satisfy

$$(2/\gamma)(n - \lambda)b_n = (b_{n-1} - b_{n+1}),\quad b_{-1} = 0. \tag{23b}$$

This recursion relation is well known.[7] We do not examine it in detail as it is sufficient for our purposes to extract one solution. A solution to (23b) is

$$b_n = \nu I_{n-\lambda}(\gamma), \tag{24a}$$

where $I_\mu(Z)$ is a modified Bessel function of the first kind of order μ.[8] We also require $I_{-1-\lambda}(\gamma) = 0$. (This forces λ to satisfy $2k + 1 > \lambda > 2k$, where $k = 0, 1, \cdots$.) The multiplicative constant ν is obtained from the normalization condition

$$|\nu|^2 \sum_{n=0}^{\infty} I^2_{n-\lambda}(\gamma) = 1. \tag{24b}$$

In Appendix B we prove that the series in (24b) converges, that $\langle\Psi| C|\Psi\rangle = 0$, and $\langle\Psi| S|\Psi\rangle \neq 0$. Thus the desired solution of (22) for which $U_1 = \frac{1}{4}$, is

$$|\Psi\rangle = \nu \sum_{n=0}^{\infty} (-i)^n I_{n-\lambda}(\gamma)\,|n\rangle,\quad \lambda = \langle N_{\rm op}\rangle,\quad (\Delta N)^2(\Delta C)^2/\langle S\rangle^2 = \tfrac{1}{4}. \tag{25}$$

Unfortunately these states do not seem to have any physical significance.

The recursion relation (23b) is also solved by the number states. These however do not minimize U_1, as we demonstrate explicitly in Appendix A.

It is clear that states which allow U_2 to reach $\frac{1}{4}$ can also be constructed.

E. We now examine the symmetric uncertainty product U_3. We first show that no states exist for which U_3 attains its minimum value of $\frac{1}{4}$. According to (3a) we have

$$\begin{aligned}(\Delta N)^2(\Delta C)^2 &= \tfrac{1}{4}\langle S\rangle^2 + \tfrac{1}{4}Q_1(\Psi) + R_1(\Psi),\\ (\Delta N)^2(\Delta S)^2 &= \tfrac{1}{4}\langle C\rangle^2 + \tfrac{1}{4}Q_2(\Psi) + R_2(\Psi).\end{aligned} \tag{26a}$$

Therefore for U_3 to be $\frac{1}{4}$ we must have

$$0 = U_3 - \tfrac{1}{4} = [\langle S\rangle^2 + \langle C\rangle^2]^{-1}[\tfrac{1}{4}Q_1(\Psi) + \tfrac{1}{4}Q_2(\Psi) + R_1(\Psi) + R_2(\Psi)]. \tag{26b}$$

Since each term on the right-hand side is positive semidefinite, $Q_{1,2}$ and $R_{1,2}$ must vanish separately, which according to (5b) requires

$$\begin{aligned}[N_{\rm op} + i\gamma_1 C]\,|\Psi\rangle &= \lambda_1\,|\Psi\rangle,\\ [N_{\rm op} + i\gamma_2 S]\,|\Psi\rangle &= \lambda_2\,|\Psi\rangle,\end{aligned} \tag{26c}$$

where γ_1 and γ_2 are real and nonzero. Evidently the commutator of the operators appearing on the left-hand side of (26c) must annihilate the state $|\Psi\rangle$. This sets the condition

$$\left[\frac{1}{\gamma_2}S + \frac{1}{\gamma_1}C + \frac{i}{2}P\right]|\Psi\rangle = 0. \tag{26d}$$

Equation (26c) may be used again to evaluate $S|\Psi\rangle$ and $C|\Psi\rangle$. Therefore (26d) becomes

$$[\gamma_1^{-2}(N_{\rm op} - \lambda_1) + \gamma_2^{-2}(N_{\rm op} - \lambda_2) + \tfrac{1}{2}P]\,|\Psi\rangle = 0. \tag{26e}$$

Expanding $|\Psi\rangle$ in number states yields the conditions

$$\begin{aligned}\left[-\frac{\lambda_1}{\gamma_1^2} - \frac{\lambda_2}{\gamma_2^2} + \frac{1}{2}\right]a_0 &= 0,\\ [\gamma_1^{-2}(n - \lambda_1) + \gamma_2^{-2}(n - \lambda_2)]a_n &= 0.\end{aligned} \tag{26f}$$

These recursion relations are solved only by the number state $|\lambda\rangle$, where $\lambda_1 = \lambda_2 = \lambda =$ integer; $\langle C\rangle = \langle S\rangle = 0$. We demonstrate in Appendix A that the number states do not minimize U_3.

Thus the direct method does not yield any solutions, and we are led to consider U_3 by the analytic method. The procedure to follow is the same as for U_1. The variation principle gives an EL equation which represents a necessary condition which must be satisfied if U_3 is to be minimized. With this condition, it can easily be verified that the coherent states are not critical states and do not minimize U_3. Again it is found that, for large N, the coherent states approximately satisfy the necessary condition, but now no restriction is set on φ. The EL equation is too complicated to serve to determine the states that do minimize U_3; hence we do not present the details of this calculation. (The recursion relation which follows from the EL equation actually is elementary, but the imposition of the subsidiary conditions is complicated. In any case the solution, if it exists, surely has no physical significance.)

Since the first variation of U_3 does not vanish for coherent states, there exist states, arbitrarily close to the coherent states, for which U_3 is smaller than it is when evaluated with coherent states. For example, with the state $|\Psi_1\rangle = \omega[|N\varphi\rangle + \epsilon e^{-\frac{1}{2}N}\,|0\rangle]$, where ω is a normalization factor, ϵ a positive small parameter, and N large, $U_3(\Psi_1)$ is smaller by an amount $2\epsilon N^{\frac{3}{2}}e^{-N}$ than with the coherent state of the same excitation.

IV. SINE-COSINE UNCERTAINTY PRODUCT

Since S and C do not commute, limitations exist on the simultaneous measurement of these two quantities. However in the classical limit these limitations

[7] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, London, 1952), p. 294.
[8] Reference 7, p. 172.

must disappear. Thus we are led to consider the uncertainty product

$$U_4 = (\Delta S)^2(\Delta C)^2. \tag{27}$$

In their original discussion of the S and C operators, SG demonstrated explicitly that there exist *unnormalizable* states

$$|\theta\rangle = \sum_{n=0}^{\infty} e^{in\theta}|n\rangle \tag{28}$$

for which $(\Delta S)^2 = 0 = (\Delta C)^2$, hence $U_4 = 0$. Carruthers and Nieto have shown that for the normalizable coherent states U_4 goes rapidly to zero for large N, and to $\frac{1}{16}$ for small N.

We now wish to use our analytic method to investigate whether there exist normalizable states which minimize U_4, and whether the coherent states are critical states for U_4. We find that no normalizable critical states exist.

To establish this result, we use (8d)

$$\left[\frac{\hat{S}^2}{(\Delta S)^2} + \frac{\hat{C}^2}{(\Delta C)^2} - 2\right]|\Psi\rangle = 0. \tag{29a}$$

For simplicity we confine ourselves to the symmetric case

$$\langle S\rangle = \langle C\rangle, \quad \langle S^2\rangle = \langle C^2\rangle.$$

Thus we need to solve

$$[(S-\alpha)^2/a^2 + (C-\alpha)^2/a^2 - 2]|\Psi\rangle = 0, \tag{29b}$$

with the subsidiary conditions

$$\begin{aligned}\langle\Psi|\Psi\rangle = 1, \quad \alpha = \langle S\rangle = \langle C\rangle,\\ a^2 = \langle S^2\rangle - \alpha^2 = \langle C^2\rangle - \alpha^2.\end{aligned} \tag{29c}$$

Equation (29b) may be simplified into

$$[v - \tfrac{1}{2}P - ue^{\frac{1}{4}i\pi}E_+ - ue^{-\frac{1}{4}i\pi}E_-]|\Psi\rangle = 0, \tag{29d}$$

where

$$\begin{aligned} u &= \sqrt{2}\alpha,\\ v &= 1 + u^2 - 2a^2.\end{aligned} \tag{29e}$$

Expanding in number states gives the recursions

$$(v - \tfrac{1}{2})a_0 - ue^{-\frac{1}{4}i\pi}a_1 = 0, \tag{30a}$$

$$va_n - u(e^{\frac{1}{4}i\pi}a_{n-1} + e^{-\frac{1}{4}i\pi}a_{n+1}) = 0, \quad n \geq 1. \tag{30b}$$

This is obviously not satisfied by the coherent states, except approximately for large N and $n \sim N$. The general solution of (30b) is given by

$$a_n = e^{i(\pi/4)n}[Ap^n + Bp^{-n}], \tag{31a}$$

$$\frac{v}{u} = p + \frac{1}{p}. \tag{31b}$$

We assume $v \neq 0$. If $|\Psi\rangle$ is to be normalizable, we must have $\sum_n |a_n|^2 = 1$; therefore A is zero if $|p| > 1$, and B is zero if $|p| < 1$. Taking the latter case and imposing (30a) and (31b) determines $p = 2u$ and sets $v = \frac{1}{2} + 2u^2$. The normalization can now be determined, and we obtain as a solution

$$|\Psi\rangle = (1-4n^2)^{\frac{1}{2}}\sum_{n=0}^{\infty} e^{i(\pi/4)n}(2n)^n|n\rangle, \quad 4u^2 < 1. \tag{32a}$$

Imposing now the subsidiary condition

$$u/\sqrt{2} = \alpha = \langle\Psi|C|\Psi\rangle = \sqrt{2}u \tag{32b}$$

gives $u = 0$ and no nontrivial normalizable solution is obtained.

Similarly when $4u^2 > 1$, no normalizable solution is obtained. Therefore we conclude that U_4 cannot be minimized by normalizable states; and only the unnormalizable states (28) minimize U_4. For these states $\langle N_{\mathrm{op}}\rangle$ diverges and they obviously represent the high excitation limit. [Such states cannot be determined by the EL equation (29d) since that equation was derived under the assumption that the solutions are normalizable.]

V. SUMMARY

In conclusion, we summarize our results. We have developed new variational techniques for the determination of states that minimize the uncertainty product of operators. By the use of these techniques we have demonstrated that the coherent states do not minimize the various uncertainty relations which can be given for number and phase operators. Although normalizable states do exist that minimize some of the number–phase uncertainty products, we do not believe that these states have any physical significance as they are strongly dependent on the specific form of the uncertainty product. Moreover, the coherent states do not even make any of the uncertainty products stationary. Thus the coherent states have no unique relevance to the classical limit of the phase operators. Indeed any state, which, when expanded in number states $|n\rangle$, has expansion coefficients a_n, which for large $\langle N_{\mathrm{op}}\rangle$ are strongly peaked and constant at $n \sim \langle N_{\mathrm{op}}\rangle$, serves to minimize approximately the uncertainty products. An example of such a state was given at the end of Sec. IIIE.

APPENDIX A

Throughout our analysis we have ignored the fact that the number states are solutions to some of the various equations we studied. We examine here whether these eigenstates of N_{op} minimize the various uncertainty products U_1, U_2, U_3 [Eq. (18)].

For number states, of course, $(\Delta N)^2 = 0$, $\langle S\rangle^2 = 0 = \langle C\rangle^2$, while $(\Delta S)^2 \neq 0 \neq (\Delta C)^2$. Therefore the uncertainty products have the indeterminate form 0/0. To obtain the value for this we proceed as follows. Consider the *excited coherent state*

$$|N\varphi n\rangle = (E_+)^n\,|N\varphi\rangle. \tag{A1}$$

These states are normalized and the number states $|n\rangle$ are reached as $N \to 0$ in these states. We therefore evaluate all matrix elements with the states $|N\varphi n\rangle$ and then let $N \to 0$.

The relevant matrix elements are easily evaluated. One finds

$$\begin{aligned} \langle N\varphi n|\,N_{\text{op}}\,|N\varphi n\rangle &= N + n,\\ \langle N\varphi n|\,N_{\text{op}}^2\,|N\varphi n\rangle &= (N+n)^2 + N,\\ \langle N\varphi n|\,T\,|N\varphi n\rangle &= \langle N\varphi|\,T\,|N\varphi\rangle, \end{aligned} \tag{A2}$$

where T is any of the operators S, S^2, C, C^2. From these it follows that

$$U_i(N\varphi n) = U_i(N\varphi) \tag{A3}$$

and

$$U_i(n) = \lim_{N\to 0} U_i(N\varphi). \tag{A4}$$

According to the definitions of the U_i's [Eq. (18)] and using the formulas (17) for the matrix elements of the S and C operators between coherent states, we have

$$\begin{aligned} U_1(N\varphi) &= N[I^2(N)\sin^2\varphi]^{-1}\\ &\quad\times[\tfrac{1}{2} - \tfrac{1}{4}e^{-N} + \tfrac{1}{2}(\cos^2\varphi - \sin^2\varphi)\\ &\quad\times J(N) - I^2(N)\cos^2\varphi],\\ U_2(N\varphi) &= U_1(\tfrac{1}{2}N\pi - \varphi),\\ U_3(N\varphi) &= U_1(\tfrac{1}{4}N\pi). \end{aligned} \tag{A5}$$

For small N, $I^2 \sim N$ and $J \sim N/\sqrt{2}$. Therefore

$$\begin{aligned} U_1(n) &= 1/4\sin^2\varphi,\\ U_2(n) &= 1/4\cos^2\varphi,\\ U_3(n) &= \tfrac{1}{2}. \end{aligned} \tag{A6}$$

It is seen that $U_1(N\varphi n)$ and $U_2(N\varphi n)$ do not approach a unique limit, viz., a limit independent of φ; therefore $U_1(n)$ and $U_2(n)$ do not exist. For U_3 we conclude that either $U_3(n) = \frac{1}{2}$, or if there exist other ways of approaching the number states, leading to a different value of $U_3(n)$, the limit does not exist. In any case, the number states do not minimize the uncertainty products.

APPENDIX B

We wish to prove that the state $|\Psi\rangle$,

$$|\Psi\rangle = \nu\sum_{n=0}^{\infty}(-i)^n I_{n-\lambda}(\gamma)\,|n\rangle, \tag{B1}$$

is normalizable, viz.,

$$\sum_{n=0}^{\infty} I_{n-\lambda}^2(\gamma) < \infty. \tag{B2}$$

For large enough n, $n-\lambda$ is positive and the following integral representation for I_μ is valid[8]:

$$I_\mu(Z) = \frac{(\frac{1}{2}Z)^\mu}{\pi^{\frac{1}{2}}\Gamma(\mu+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\mu-\frac{1}{2}}e^{\pm 2t}\,dt;\quad \operatorname{Re}\mu > -\tfrac{1}{2}. \tag{B3}$$

Evidently

$$I_{n-\lambda}^2(\gamma) \le \frac{(\frac{1}{4}\gamma^2)^{n-\lambda}}{[\Gamma(n-\lambda+\frac{1}{2})]^2}M(\gamma),\quad n > \lambda - \tfrac{1}{2}, \tag{B4}$$

where M is positive and independent of n and λ. Therefore the series (B2) converges.

We also need to show that $\langle\Psi|\,C\,|\Psi\rangle = 0$. This is readily established from the formulas (16a) and (B1).

Finally we establish that $\langle\Psi|\,S\,|\Psi\rangle \neq 0$. Recall that S is proportional to the commutator of N_{op} and C. According to the general discussion of Sec. IIC, we know that the expectation value of the commutator is proportional to $\gamma(\Delta C)^2$. Since γ is nonzero, we may prove that $\langle S\rangle$ is nonzero by showing that $(\Delta C)^2$ does not vanish. However $(\Delta C)^2 = \langle C^2\rangle$ since $\langle C\rangle = 0$. But $\langle C^2\rangle$ is nonzero since the operator C manifestly does annihilate $|\Psi\rangle$.