IOP Publishing Physica Scripta
Phys. Scr. **78** (2008) 035401 (10pp) doi:10.1088/0031-8949/78/03/035401

On SU(1,1) intelligent coherent states

G M Abd Al-Kader and A-S F Obada

Mathematics Department, Faculty of Science, Al-Azhar University, Nasr City 11884, Cairo, Egypt

E-mail: abdalkaderg@sci-azhar.edu.eg

Received 11 May 2007
Accepted for publication 9 July 2008
Published 22 August 2008
Online at stacks.iop.org/PhysScr/78/035401

Abstract
Generalized coherent states associated with SU(1,1) Lie algebra are reviewed. A state is called intelligent if it satisfies the strict equality in the Heisenberg uncertainty relation. The eigenvalue problem satisfied by intelligent states (IS) is solved. The IS associated with SU(1,1) Lie algebra are investigated. We have constructed some realizations for our results of IS, and some applications are discussed. Some nonclassical properties such as Glauber second-order correlation function, photon number distribution and squeezing are investigated.

PACS numbers: 42.50.Dv, 03.65.Fd, 02.20.Qs

1. Introduction

Some general structural features of dynamical theories can be exhibited in a study of the relation between classical and quantum mechanics. The essential dynamical structure of these theories as that of a Lie algebra of functions of basic dynamical variables provides the infinitesimal generators of a group of dynamical transformations. Therefore, group theory is a powerful means for joining things which, at first glance, seemed to be completely disjoint. It also provided means to disentangle complicated operations, and made its way into quantum mechanics [1], chiefly into the theory of atoms, molecules and solids. The relation between classical and quantum mechanics in the framework of Lie groups has been studied [2]. The group theoretical methods have been widely used in various branches of physics, such as high-energy physics, condensed matter, atomic and nuclear physics [2–4]. Particularly, the infiltration of the group theory into quantum optics occurred later, unlike the use of the Heisenberg–Weyl group from its very beginning in studying some physical structures [1]. Groups involving simple Lie algebras such as SU(2), SU(1,1) and their simple generalizations (e.g. higher symplectic groups SP(2n,R)) have been used in studying many problems in quantum optics [3, 4].

Glauber coherent states (CSs) [5] are eigenstates of an annihilation operator with the property that the uncertainty product for the position and momentum canonical variables attains its minimum value, when the two standard variances are equal. Squeezed coherent states (SCSs) ([6–10], [11] and references therein) are characterized by the property that one of the variances is smaller than that in a CS. This happens, naturally, at the expense of the other, because of Heisenberg's principle. It may therefore prove to be useful in low-noise detection experiments. The original forms in which SCSs were written, are found in [6]. The notation of coherent and SCSs is not restricted to the electromagnetic field (characterized by the photon annihilation and creation operators a and a^+) but can be extended to any set of operators obeying a Lie algebra [12–18]. Quantum optical problems naturally offer us a number of Lie algebras such as: h(4), SU(2) and SU(1,1). The h(4) (or Heisenberg–Weyl) algebra involves the electromagnetic field operators a, a^+ and the basis for the definition of the usual CSs.

The dynamical group SU(1,1) has long been used in quantum optics as it is intimately related to the squeeze operator which is an element of the SU(1,1) group [17]. This group is the simplest non-Abelian noncompact Lie group with a simple Lie algebra, and shares with SU(2) a common complex extension. The CSs of the SU(1,1) group can be divided into two broad categories: (a) the Barut–Girardello (BGCSs) [15] and (b) the Perelomov (PCSs) [16]. In two papers [15, 16] one finds basic investigations in which the concept of CSs has been extended beyond the Heisenberg–Weyl group for the first time. Perelomov [16, 17] has generalized the idea of CSs to other Lie groups, in which elaborate methods of groups were employed in studying the properties of physical systems. The BGCSs have been investigated in mathematical framework in [19]. The duality of these two types of SU(1,1) CSs and an intermediate type have been considered in [20]. In addition, the SU(1,1) Lie algebra is of great interest, because it can characterize many kinds of optical systems. In particular, the bosonic realization of SU(1,1) describes the degenerate and non-degenerate parametric amplifiers. It would be interesting to mention here that the customary squeezed states of photons can be viewed as realizations of the SU(1,1) Lie group [21, 22].

Intelligent states (IS) are quantum states which equalize the Heisenberg uncertainty relation [23–40]. It goes without saying that the states providing an equality in the uncertainty relation do not, in general, reach minimum uncertainty [18]. In recent years, there have been many studies concerning IS, mainly in the context of quantum optics [23–40]. One of the principal reasons for this interest is the close relationship between IS and squeezing. These states may exhibit various nonclassical properties such as squeezing, and sub-Poissonian statistics. The so-called intelligent spin states have been constructed [23]; they can also be generated from the atomic CSs or Bloch states [24]. A simple algebraic method which permits the reproduction of these states has been developed for calculating the matrix elements [25] and computing the Clebsch–Gordan coefficients [26]. The existence of IS associated with the non-compact group SU(1,1) has been investigated [27]. The SU(1,1) IS are studied in the observables in two mutually orthogonal directions in a $2+1$ Minkowski space by Puri and Agarwal [28]. But Fu and Sasaki defined a class of squeezed states for the SU(1,1) Lie algebra as an approach for solving the eigenvalue problem corresponding to equality of uncertainty relation [29]. The concept of algebra eigenstates and two-photon algebra eigenstates is related to the IS; it has been used in studying squeezing [30, 31]. In addition, the two-mode IS of SU(1,1) Lie group and its statistical properties have been considered [32].

A scheme for generation of single-mode IS based on the process of non-degenerate down-conversion has been discussed in [33]. This scheme employs quantum correlations (the entanglement) created in a non-degenerate parametric amplifier between the vacuum and squeezed vacuum, which may manipulate the state of one of the modes by measurement of the photon number in the other. For the quantized vibrational motion of a trapped ion [34], the scheme for generating SU(1,1) IS has been proposed in [35]. The proposed scheme for realization of both the single- and two-mode SU(1,1) IS of centre-of-mass motion of a trapped ion [34] may be accessible to current experimental set-ups. Experimentally feasible IS are studied for parameter values leading to the crescent topography of their quasi-probability, and to increase the sensitivity of interferometric measurement [36]. An analytic representation in the unit disk for SU(1,1) IS has been performed in [37]. The IS associated with the Holstein–Primakoff realization of the SU(1,1) Lie algebra have been considered in [38]. These states contract, under certain conditions, to Glauber CSs or SCSs [38]. Recently, polynomial IS, a simple method for constructing IS, especially for SU(2) and SU(3), have been introduced [39]. Also, large-uncertainty IS for angular momentum and angle have been discussed [40], and the difference between the IS and minimum uncertainty product states is emphasized.

On the other hand, the SU(1,1) generalized IS which minimize the Robertson–Schrödinger uncertainty relation are considered mathematically by Trifonov [41]. The eigenvalue equation has been solved using the differential forms of the generators of an algebra. Some treatments in this direction related to different potentials [42–45] and different algebras [46–48] have been introduced. Thus, generalized IS have been investigated for nonlinear oscillators [42], for two-body Calogero model [43], for infinite square well potential [44] and for exact solvable quantum systems [45]. These results were obtained by using the Gazeau–Klauder and Klauder–Perelomov generalized CSs to derive the corresponding Robertson–Schrödinger IS [41–48]. In addition, the IS for an interpolating algebra [46], for SU(N) algebra [47], and SU(3) [48] have been investigated.

The aim of this work is to solve a recurrence relation corresponding to the eigenvalue equation of IS, rather than using the differential forms for generators discussed in [33, 39, 41]. This paper is organized as follows: in section 2, we briefly review some basic results of CSs associated with SU(1,1) Lie algebra. In section 3, we review the definition of IS and find special cases. The solution of the main eigenvalue problem has been introduced. Some realizations and special cases are discussed in section 4. In section 5, we discuss some statistical properties of the obtained IS such as correlation function and photon number distribution.

2. SU(1,1) CSs

In this section, some basic properties of the SU(1,1) Lie algebra CSs and of the unitary irreducible representations needed in the following sections are collected.

The SU(1,1) Lie algebra is spanned by the three generators K_1, K_2 and K_3, which satisfy the following commutation relations:

$$[K_1, K_2] = -\mathrm{i}K_3, \quad [K_2, K_3] = \mathrm{i}K_1, \quad [K_3, K_1] = \mathrm{i}K_2. \tag{2.1}$$

It is convenient to use the raising and lowering generators $K_\pm = K_1 \pm \mathrm{i}K_2$, which satisfy

$$[K_3, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_3. \tag{2.2}$$

The Casimir operator $K^2 = K_3^2 - K_1^2 - K_2^2$ for any irreducible representation is $K^2 = k(k-1)I$. Thus, a representation of SU(1,1) is determined by the parameter k which is the so-called Bergmann index. The corresponding Hilbert space is spanned by the complete orthonormal basis $|n, k\rangle$:

$$\langle m, k|n, k\rangle = \delta_{mn}, \qquad \sum_{n=0}^{\infty} |n, k\rangle\langle n, k| = I. \tag{2.3}$$

For SU(1,1) there are many unitary irreducible representations, and since SU(1,1) is a noncompact group, they are all of infinite dimensions. Some of the representations are, in fact, continuous, but here we shall only deal with the representations known as the positive discrete series for which the operator K_3 is diagonal and has a discrete spectrum. Its discrete representation is

$$\begin{aligned} K_+|n, k\rangle &= \sqrt{(n+1)(2k+n)}|n+1, k\rangle, \\ K_-|n, k\rangle &= \sqrt{n(2k+n-1)}|n-1, k\rangle, \\ K_3|n, k\rangle &= (n+k)|n, k\rangle, \end{aligned} \tag{2.4}$$

where $(n = 0, 1, 2, \ldots)$. The ground state (the cyclic vector) of the representation is given by the condition $K_-|0, k\rangle = 0$. All states can be obtained from the lowest state $|0, k\rangle$ by the action of the 'raising' operator K_+ according to

$$|m, k\rangle = \sqrt{\frac{\Gamma(2k)}{m!\Gamma(2k+m)}}(K_+)^m|0, k\rangle, \tag{2.5}$$

where $\Gamma(x)$ is the gamma function.

The PCS $|\alpha, k\rangle_{\mathrm{Per}}$ for SU(1,1) Lie algebra may be obtained by applying the unitary operator $D_{\mathrm{Per}}(\xi)$ to the lowest state $|n = 0, k\rangle$ [16, 17],

$$\begin{aligned}|\alpha, k\rangle_{\mathrm{Per}} &= D_{\mathrm{Per}}(\xi)|0, k\rangle\\ &= (1 - |\alpha|^2)^k \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(2k+n)}{n!\Gamma(2k)}}\alpha^n|n, k\rangle,\end{aligned} \tag{2.6}$$

where $\xi = |\xi|\mathrm{e}^{\mathrm{i}\theta_0}$ is a complex number, with $\alpha = \mathrm{e}^{\mathrm{i}\theta_0}\tanh|\xi|$, and

$$\begin{aligned}D_{\mathrm{Per}}(\xi) &= \exp(\xi K_+ - \xi^* K_-)\\ &= \exp(\alpha K_+)(1 - |\alpha|^2)^{K_3}\exp(-\alpha^* K_-)\end{aligned} \tag{2.7}$$

is the SU(1,1) displacement operator. The PCSs form an overcomplete set of states.

There is another CS of SU(1,1) which is known as the BGCS [15], which is defined as the eigenstate of the lowering operator K_-,

$$K_-|\alpha, k\rangle_{\mathrm{BG}} = \alpha|\alpha, k\rangle_{\mathrm{BG}}, \tag{2.8}$$

and it can be expressed as

$$|\alpha, k\rangle_{\mathrm{BG}} = \sqrt{\frac{|\alpha|^{2k-1}}{I_{2k-1}(2|\alpha|)}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!\Gamma(n+2k)}}|n, k\rangle, \tag{2.9}$$

where $I_\nu(x)$ is the modified Bessel function of the first kind. The BGCSs are normalized, but they are not orthogonal to each other.

3. SU(1,1) IS

The uncertainty relation limits the precise knowledge of conjugate physical quantities of a system. The states which minimize the uncertainty relation can describe the quantum system as precisely as possible. First, for given two self-adjoint operators A and B, one can obtain, using the Cauchy–Schwartz inequality, the uncertainty relation

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geqslant \tfrac{1}{4}|\langle[A, B]\rangle|^2, \tag{3.1}$$

where the variance and expectation value are given by $(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2$ and $\langle A\rangle = \langle\psi|A|\psi\rangle$, respectively. A state is called intelligent if it satisfies the strict equality in (3.1). It is well known [23] that such states, or IS, must satisfy the eigenvalue equation

$$(A - \mathrm{i}\lambda B)|\psi\rangle = \eta|\psi\rangle, \tag{3.2}$$

where λ is a positive real parameter and η a complex number. In the case in which $[A, B] = cI$, where c is constant and I the identity operator, the minimum uncertainty states coincide with the IS [18].

The SU(1,1) Lie algebra IS are defined as follows: let us now apply the above approach to the special case where A, B are K_1, K_2 of equation (2.1), then the uncertainty relation would be

$$\langle(\Delta K_1)^2\rangle\langle(\Delta K_2)^2\rangle \geqslant \tfrac{1}{4}\langle K_3\rangle^2. \tag{3.3}$$

Accordingly, a state is said to be SU(1,1)-squeezed [18] if

$$\langle(\Delta K_i)^2\rangle \leqslant \tfrac{1}{2}|\langle K_3\rangle|, \quad i = 1, 2. \tag{3.4}$$

However, IS $|\psi\rangle$ are solutions of the eigenvalue problem

$$(K_1 - \mathrm{i}\lambda K_2)|\psi\rangle = \eta|\psi\rangle. \tag{3.5}$$

It is convenient to rewrite equation (3.5) in terms of $K_\pm$ as

$$(\alpha_1 K_- + \beta_1 K_+)|\psi\rangle = 2\eta|\psi\rangle, \tag{3.6}$$

where $\alpha_1 = 1 + \lambda$ and $\beta_1 = 1 - \lambda$. In order to have normalizable solutions, it is necessary that $\alpha_1 > \beta_1$. It is clear that, for the special case $\lambda = 1$, equation (3.6) reduces to BGCSs. Gerry and Grobe [38] proved that in the limit of large k the SU(1,1) Lie algebra contracts to that of the Heisenberg–Weyl algebra and the Holstein–Primakoff SU(1,1) CSs contract to ordinary CSs or SCSs, depending on λ.

Let us now consider the eigenvalue problem (3.6), we expand the state $|\psi\rangle$ on the basis of $|n, k\rangle$, as

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n(k)|n, k\rangle \tag{3.7}$$

and apply (2.4) to obtain the recurrence relation among the coefficients c_n as follows:

$$\begin{aligned}&\alpha_1\sqrt{(m+1)(m+2k)}c_{m+1} + \beta_1\sqrt{m(m+2k-1)}c_{m-1} - 2\eta c_m\\ &\quad = 0.\end{aligned} \tag{3.8}$$

Assuming $c_m = (\sqrt{\beta_1/\alpha_1})^m d_m$, we obtain

$$\begin{aligned}&\tfrac{1}{2}\sqrt{(m+1)(m+2k)}d_{m+1} + \tfrac{1}{2}\sqrt{m(m+2k-1)}d_{m-1} - zd_m\\ &\quad = 0,\end{aligned} \tag{3.9}$$

where $z = \frac{\eta}{\sqrt{\alpha_1\beta_1}}$. Comparing (3.9) with the Pollaczek polynomials $P_n(\theta, b)$ [49–51], the recurrence relation [50], namely,

$$\Delta_{n-1}P_{n-1}(\theta, b) + \Delta_n P_{n+1}(\theta, b) = \theta P_n(\theta, b); \tag{3.10}$$

with

$$\Delta_n = \tfrac{1}{2}\sqrt{(n+1)(n+2b)}; \quad \text{and} \quad P_0(\theta, b) = 1,$$

the solution for the eigenvalue equation (3.9) is directly the Pollaczek polynomials, namely,

$$\begin{aligned}d_m &= P_m(z, k)\\ &= (\mathrm{i})^m\left(\frac{\Gamma[m+2k]}{m!\Gamma[2k]}\right)^{m/2} {}_2F_1(-m, k+\mathrm{i}z; 2k; 2),\end{aligned} \tag{3.11a}$$

which for every $b > 0$ form a complete orthonormal set of polynomials on the real line with the weight function

$$\rho_b(\theta) = (2)^{2b-1}[|\Gamma[b+\mathrm{i}\theta]|^2/(\pi\Gamma[2b])]. \qquad (3.11b)$$

The orthonormality and completeness of the set of polynomials $P_n(\theta, b)$ using an integral representation of the hypergeometric function and some useful results of Pollaczek polynomials can be found in [50]. We thus obtain the final result for the IS in the form

$$|\psi\rangle = c_0 \sum_{m=0}^{\infty} (\mathrm{i})^m \left(\frac{(1-\lambda)\Gamma[m+2k]}{m!(1+\lambda)\Gamma[2k]} \right)^{m/2} \times {}_2F_1(-m, k+\mathrm{i}z; 2k; 2)|m, k\rangle, \qquad (3.12a)$$

where the normalization factor c_0 has the form

$$|c_0|^{-2} = \left\{ \sum_{m=0}^{\infty} \left(\frac{(1-\lambda)\Gamma[m+2k]}{m!(1+\lambda)\Gamma[2k]} \right)^m \times |{}_2F_1(-m, k+\mathrm{i}z; 2k; 2)|^2 \right\} \qquad (3.12b)$$

and${}_2F_1(a, b; c; x)$ is the hypergeometric function and $z = \frac{\eta}{\sqrt{(1+\lambda)(1-\lambda)}}$. After the review and formulation of the problem, we introduced the analytical solution for the recursion relation equation (3.9) by a new approach different from that used in [33] for IS, and [41] for generalized IS, which minimize the Robertson–Schrödinger uncertainty relation. The solution for the IS of the SU(1,1) group given by equation (3.12) is convenient for studying some realizations, special cases and some nonclassical properties of the field states.

4. Some realizations for SU(1,1) and special cases of IS

4.1. The one-mode realization

In the one-mode realization for SU(1,1) algebra, we form the quadratic combinations

$$K_+ = \tfrac{1}{2}a^{+2}, \quad K_- = \tfrac{1}{2}a^2, \quad K_3 = \tfrac{1}{2}(a^+a + \tfrac{1}{2}). \qquad (4.1)$$

The Casimir operator is $K^2 = -3/16I$. Therefore, there are two irreducible representations with $k = 1/4$ and $k = 3/4$; consequently, we obtain a realization of SU(1,1) algebra. In this case, the corresponding eigenvalue equation is a special case from (3.6) of IS, which reads as

$$(\alpha_1 a^2 + \beta_1 a^{+2})|\psi\rangle = 4\eta|\psi\rangle. \qquad (4.2)$$

The minimum uncertainty states for amplitude-squared squeezing [51–53] are the solutions to the eigenvalue problem (4.2). These states have been studied and the general solutions to the eigenvalue have been found [53]. Also, these states are associated with the case of $j = 2$ in the definition of higher power CSs [50, 54]. Complete solutions of (4.2) in the coordinate representation, in terms of confluent hypergeometric functions have been introduced [54]. The eigenstates of the operator $K_1 = 1/2(a^2 + a^{+2})$, i.e. $\lambda = 0$, have been discussed in [50]. Most physical states are produced for some values of λ, when $0 < \lambda < 1$ and $\lambda > 1$.

As a special case of (4.2), in the two-photon realization the Barut–Girardello eigenvalue equation takes the form

$$a^2|\alpha, k\rangle_{\mathrm{BG}} = 2\alpha|\alpha, k\rangle_{\mathrm{BG}}. \qquad (4.3)$$

Therefore, it is clear that the Barut–Girardello states coincide with the even and odd CS [55, 56],

$$|\alpha, 1/4\rangle = |\alpha\rangle_e = \frac{1}{\sqrt{2(1+\mathrm{e}^{-2|\alpha|^2})}}[|\alpha\rangle + |-\alpha\rangle],$$
$$|\alpha, 3/4\rangle = |\alpha\rangle_o = \frac{1}{\sqrt{2(1-\mathrm{e}^{-2|\alpha|^2})}}[|\alpha\rangle - |-\alpha\rangle] \qquad (4.4)$$

for $k = 1/4$ and $k = 3/4$, respectively. Here $|\alpha\rangle = D(\alpha)|0\rangle$ are the Glauber CS, e and o indicate even ($k = 1/4$) and odd ($k = 3/4$) subspaces correspondingly.

4.2. Perelomov CSs

All the PCSs $|\alpha, k\rangle_{\mathrm{per}}$ can be viewed as the SU(1,1) IS. They are eigenstates of the following eigenvalue equation [29]:

$$(K_- - \alpha^2 K_+)|\alpha, k\rangle_{\mathrm{per}} = 2k\alpha|\alpha, k\rangle_{\mathrm{per}}. \qquad (4.5)$$

This can be directly proved by differentiating $|\alpha, k\rangle_{\mathrm{per}}$ with respect to $|\xi|$. Note that if $\alpha^2 = -\frac{\beta_1}{\alpha_1}$, the equation (4.5) tends to equation (3.6) with eigenvalue $2k\alpha_1\alpha$ [29]. Here, we can show some realizations of PCSs towards nonclassical quantum states. For one-mode realization (equation (4.1)) the PCSs are the squeezed vacuum states [6]. In this case, the SU(1,1) CSs are the single-mode squeezed states. For $k = 1/4$, the squeezed vacuum is given by

$$|\alpha, 1/4\rangle = (1-|\alpha|^2)^{1/4} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} \alpha^n |2n\rangle \qquad (4.6a)$$

and for $k = 3/4$, the squeezed one-photon state is given by

$$|\alpha, 3/4\rangle = (1-|\alpha|^2)^{3/4} \sum_{n=0}^{\infty} \frac{\sqrt{(2n+1)!}}{2^n n!} \alpha^n |2n+1\rangle \qquad (4.6b)$$

with $\alpha = \frac{\xi}{|\xi|} \tanh|\xi| = \tanh r\, \mathrm{e}^{\mathrm{i}\theta_0}$.

4.3. Nonlinear squeezed states (NLSSs)

The nonlinear coherent states (NLCSs) [57–59] $|\alpha\rangle_f$, are right-hand eigenstates of the product of the boson annihilation operator a and the operator-valued function $f(a^+a)$ of the number operator $N = a^+a$. They satisfy $A|\alpha\rangle_f = \alpha|\alpha\rangle_f$, where $A = af(N)$. The nature of the nonlinearity depends on the choice of the function $f(N)$. These states may appear as stationary states of the center-of-mass motion of a trapped and laser-driven ion far from the Lamb–Dicke regime [57], and may be considered as particular cases of f-CSs [58]. In a similar context of NLCSs, the nonlinear squeezed vacuum states are given by

$$|z\rangle_f = \exp\left[\tfrac{1}{2}(zA^{+2} - z^*A^2)\right]|0\rangle = S_{\mathrm{f}}(z)|0\rangle$$

for the operator function $f(N)$ being the unitary operator in this definition, i.e. $f^+ = f^{-1}$ and $[A, A^+] = 1$ [60–62]. Hence, in what follows, the two cases in which $f(N)$ is the unitary or the non-unitary operator function will be discussed.

4.3.1. The unitary nonlinear function. The NLSSs [54] realization of SU(1,1) Lie group has been constructed [61, 62]. We mention the NLSSs realization of the SU(1,1) group by constructing the K_+ and K_- operators in the following way [61, 62]:

$$K_+ = \tfrac{1}{2}(f(N)^+a^+)^2 = \tfrac{1}{2}A^{+2},\qquad K_- = \tfrac{1}{2}(af(N))^2 = \tfrac{1}{2}A^2, \tag{4.7a}$$

where the operator-valued function $f(N)$ is a reasonably behaved function of the photon number operator N. For the operator K_3 to be in the form of (4.1), f must be a unitary operator, i.e. $f^+ = f^{-1}$. Under this condition, we get

$$K_3 = \tfrac{1}{2}(N + \tfrac{1}{2}). \tag{4.7b}$$

The unitary group operator $D_{\mathrm{Per}}(\alpha)_f$ for the nonlinear squeezing case is the operator given in (2.6), but with $K_\pm, K_3$ given by (4.7) under the condition $f^+ = f^{-1}$. Therefore, the SU(1,1) CSs are the NLSSs. Consequently, the nonlinear-squeezed vacuum is given by

$$|\alpha, 1/4\rangle_f = (1-|\alpha|^2)^{1/4}\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}(f(2n)!)^{-1}}{2^n n!}\alpha^n|2n\rangle, \tag{4.8a}$$

whereas the nonlinear-squeezed one-photon state is given by

$$|\alpha, 3/4\rangle_f = (1-|\alpha|^2)^{3/4}\sum_{n=0}^{\infty}\frac{\sqrt{(2n+1)!}}{2^n n!(f(2n+1)!)}\alpha^n|2n+1\rangle, \tag{4.8b}$$

where $f(0)=1$, $f(n)! = \prod_{i=0}^{n} f(i)$. The states (4.8) are the SU(1,1) group realizations by a nonlinear-squeezed vacuum and one-photon states.

4.3.2. Realization for non-unitary f. Even if the operator function $f(N)$ is not a unitary operator, one can still define a NLSS as given in [61, 62]. The steps towards this depends on using a canonical conjugate operator. If we have

$$A = af(N),\qquad A^+ = [f(N)]^+a^+, \tag{4.9a}$$

then the canonical conjugate operators are

$$B^+ = \frac{1}{f(N)}a^+,\qquad B = a\frac{1}{[f(N)]^+}. \tag{4.9b}$$

The operators A and B satisfy the commutation relations

$$[A, B^+] = 1,\qquad [B, A^+] = 1. \tag{4.9c}$$

In what follows, the operator-valued function f is assumed to be a well-behaved real function. The use of the operators A and B^+ (instead of A^+) does not ensure the squeeze operator being unitary; thus, one looks for the eigenfunctions of the operator

$$C_1 = \frac{1}{\sqrt{1-|\xi_1|^2}}(A - \xi_1 B^+), \tag{4.10}$$

with the eigenvalue zero, i.e. the nonlinear-squeezed vacuum states are the solutions of the equations

$$C_1|\Psi_1\rangle_f = 0. \tag{4.11}$$

It is effortless to find the expression

$$|\Psi_1\rangle_f = N_1\sum_{m=0}^{\infty}\frac{\sqrt{(2m)!}(f(2m)!)^{-1}}{2^m m!}\xi_1^m|2m\rangle, \tag{4.12a}$$

where N_1 is the normalization constant, while the nonlinear-squeezed one-photon states are the solutions of the eigenvalue equation

$$C_1^2|\Phi_1\rangle = 0. \tag{4.13}$$

Carrying out the calculations, it is easy to find that these states are composed of the odd Fock states and are given by

$$|\Phi_1\rangle_f = \acute{N}_1\sum_{m=0}^{\infty}\frac{\sqrt{(2m+1)!}(f(2m+1)!)^{-1}}{2^m m!}\alpha_1^m|2m+1\rangle, \tag{4.14}$$

with $\acute{N}_1$ normalization constant. Some of the nonclassical properties of these states have been discussed recently [61, 62]. The states for the particular case of even and odd states of single-mode realization of SU(1,1) are obtained by letting $k = 1/4$ and $k = 3/4$, respectively. Therefore, the basis $|m, 1/4\rangle$ in terms of Fock state basis is $|2m\rangle$ and $|m, 3/4\rangle = |2m+1\rangle$ [63, 64].

4.4. The two-mode realization for SU(1,1) IS

Let a (a^+) and b (b^+) represent the usual bosonic operators associated with the two field modes. The generators can be constructed [32, 63] as follows:

$$K_+ = a^+b^+,\qquad K_- = ab,\qquad K_3 = \tfrac{1}{2}(a^+a + b^+b) + \tfrac{1}{2}. \tag{4.15}$$

Consequently, the quadrature operators are given by

$$K_1 = \tfrac{1}{2}(K_+ + K_-) = \tfrac{1}{2}(a^+b^+ + ab),\qquad K_2 = \frac{1}{2\mathrm{i}}(K_+ - K_-) = \frac{1}{2\mathrm{i}}(a^+b^+ - ab). \tag{4.16}$$

The eigenvalue problem equation (3.6) in terms of bosonic operators of the two modes has the form

$$(\alpha_1 ab + \beta_1 a^+b^+)|\psi\rangle = \eta|\psi\rangle. \tag{4.17}$$

The general solution for the equation (4.17) has been investigated in [36, 63]. To obtain a solution by using the two-mode Fock states as a basis, equation (4.17) reduces to the same form of the recurrence relation (3.9), with $k = \frac{1}{2}(N_a - N_b + 1)$.

Note that for $\beta_1 = 0$, the eigenvalue problem reduces to

$$ab|\psi\rangle = \eta|\psi\rangle, \tag{4.18}$$

whose solution is the pair CSs [64], or correlated SU(1,1) CSs [65]. These states may be produced by the action of a nondegenerate parametric amplifier on a two-mode state [65].

4.5. Nonlinear two-mode realization for SU(1,1) IS

The generators for the unitary operator function f_i such that $f_i^+ = f_i^{-1}$ and $i = 1, 2$, can be constructed as follows:

$$
\begin{aligned}
K_+ &= f_1(N_a)a^+ f_2(N_b)b^+, \\
K_- &= a\frac{1}{f_1(N_a)} b\frac{1}{f_2(N_b)}, \\
K_3 &= \tfrac{1}{2}(N_a + N_b) + \tfrac{1}{2},
\end{aligned}
\tag{4.19}
$$

where $a^+a = N_a$ and $b^+b = N_b$. Then, the original commutation relations of SU(1,1) Lie algebra hold. However, it is to be noted that K_+ is the hermitian conjugate of K_- in this case, because $f_i^+ = f_i^{-1}$ and $i = 1, 2$. The eigenvalue problem equation (3.6) in terms of nonlinear bosonic operators of the two modes has the form

$$
\left\{\alpha_1\left(a\frac{1}{f_1(N_a)} b\frac{1}{f_2(N_b)}\right) + \beta_1(f_1(N_a)a^+ f_2(N_b)b^+)\right\}|\psi\rangle = \eta|\psi\rangle. \tag{4.20}
$$

The above are very important for new developments in quantum information processing. The generation and some statistical properties of a special case of equation (4.20), the so-called nonlinear pair-CSs, have been investigated recently [66]. In this case we set $\beta_1 = 0$, then the eigenvalue problem (4.20) reduces to

$$
a\frac{1}{f_1(N_a)} b\frac{1}{f_2(N_b)}|\psi\rangle = \eta|\psi\rangle, \tag{4.21}
$$

which is the two-mode nonlinear CSs studied in [67].

5. Nonclassical properties

Nonclassical effects are characterized by photon anti-bunching, sub-Poissonian photon statistics and quadrature squeezing (see for example [68]). The definition of nonclassicality is based on the existence of a well-behaved P-function [5, 69]. This means that a state is considered to have a classical counterpart if the P-function has the properties of a probability measure. For a nonclassical state, it fails to be interpreted as a probability. Some methods for the characterization of the nonclassical properties of radiation have also been discussed [69–72]. For example, the negativity of the Wigner function may be used as a signature of nonclassicality [73].

In this section, we shall examine the auto-correlation function, photon number distribution and squeezing. For simplicity we set $c_n(k) = c_n$ of equations (3.7) and (3.12) in our calculations. It is clear that the coefficients c_n in general depend on polynomials; therefore, we shall resort to performing numerical calculations. In general, for special values of the Bergmann index k and numerical values of λ and η, the IS may correspond to one of the bosonic states.

5.1. Auto-correlation function $g^{(2)}(0)$

To calculate the moments of the quadratures (4.16) in the IS, one has to find average values of products of the operators K_- and K_+ in these states, they have the following forms:

$$
\langle K_-^q\rangle = \sum_{r=0}^{\infty} c_r c_{r-q}^* \sqrt{\frac{r!(2k+r-1)!}{(r-q)!(2k+r-q-1)!}}, \tag{5.1}
$$

$$
\langle K_+^p\rangle = \sum_{r=0}^{\infty} c_r c_{r+p}^* \sqrt{\frac{(r+p)!(2k+r+p-1)!}{r!(2k+r-1)!}}, \tag{5.2}
$$

$$
\begin{aligned}
\langle K_+^p K_-^q\rangle = \sum_{r=0}^{\infty} c_r c_{r+p-q}^* &\sqrt{\frac{r!(2k+r-1)!}{(r-q)!(2k+r-q-1)!}} \\
&\times \sqrt{\frac{(r+p-q)!(2k+r+p-q-1)!}{(r-q)!(2k+r-q-1)!}},
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
\langle K_-^q K_+^p\rangle = \sum_{r=0}^{\infty} c_r c_{r+p-q}^* &\sqrt{\frac{(r+p)!(2k+r+p-1)!}{(r)!(2k+r-1)!}} \\
&\times \sqrt{\frac{(r+p)!(2k+r+p-1)!}{(r+p-q)!(2k+r+p-q-1)!}},
\end{aligned}
\tag{5.4}
$$

where c_n are given in equation (3.12).

The second-order coherence function for the SU(1,1) operators is defined by [74]

$$
g^{(2)} = \frac{\langle K_+{}^2 K_-^2\rangle}{\langle K_+ K_-\rangle^2}. \tag{5.5}
$$

A method for measuring general space–time-dependent correlation functions of quantized radiation fields has been proposed in [73]. It is shown that all the required moments can be determined by homodyne correlation measurements [72]. The second-order correlation function of the SU(1,1) realization in the BGCSs and PCSs is also considered [74]. In the one-mode realization (4.1), the Glauber second-order coherence function [5] can be obtained. The light with $g^{(2)} < 1$ is sub-Poissonian light, with $1 < g^{(2)} < 2$ is super-Poissonian light, and with $g^{(2)} > 2$ is called super-thermal light. Coherent light has $g^{(2)} = 1$, whereas thermal light has $g^{(2)} = 2$.

In figure 1, we display the behaviour of the auto-correlation function $g^{(2)}$ against λ (figures 1(a) and (b)) and against η (figures 1(c) and (d)). We assume the other parameters as follows: $\eta = 1/4, 1$ and $k = 1/4, 3/4$, in figures 1(a) and (b), and $\lambda = 1/4, 1$ and $k = 1/4, 3/4$, in figures 1(c) and (d).

For the case of $\eta = 1/4$, we note that the state with $k = 1/4$ starts with sub-Poissonian behaviour for a long range of λ and becomes super-Poissonian for $\lambda > 2.3$. On the other hand, the state $k = 3/4$ starts at slightly super-Poissonian, decreases to sub-Poissonian until $\lambda = 0.9$ at which it reaches its minimum and then increases very fast to become super-Poissonian around $\lambda > 1.2$, and acquires super-thermal behaviour for values of $\lambda > 1.5$.

In figures 1(c) and (d), we plot the autocorrelation function $g^{(2)}$ against η. It is found that super-Poissonian behaviour is apparent for a short range of η, when we take $\lambda = 1/4$ for both $k = 1/4$ and $k = 3/4$. This range extends as λ is increased to 1, see figures 1(c) and (d). As seen in figure 1(d), sub-Poissonian behaviour exists for the rest of the range considered.

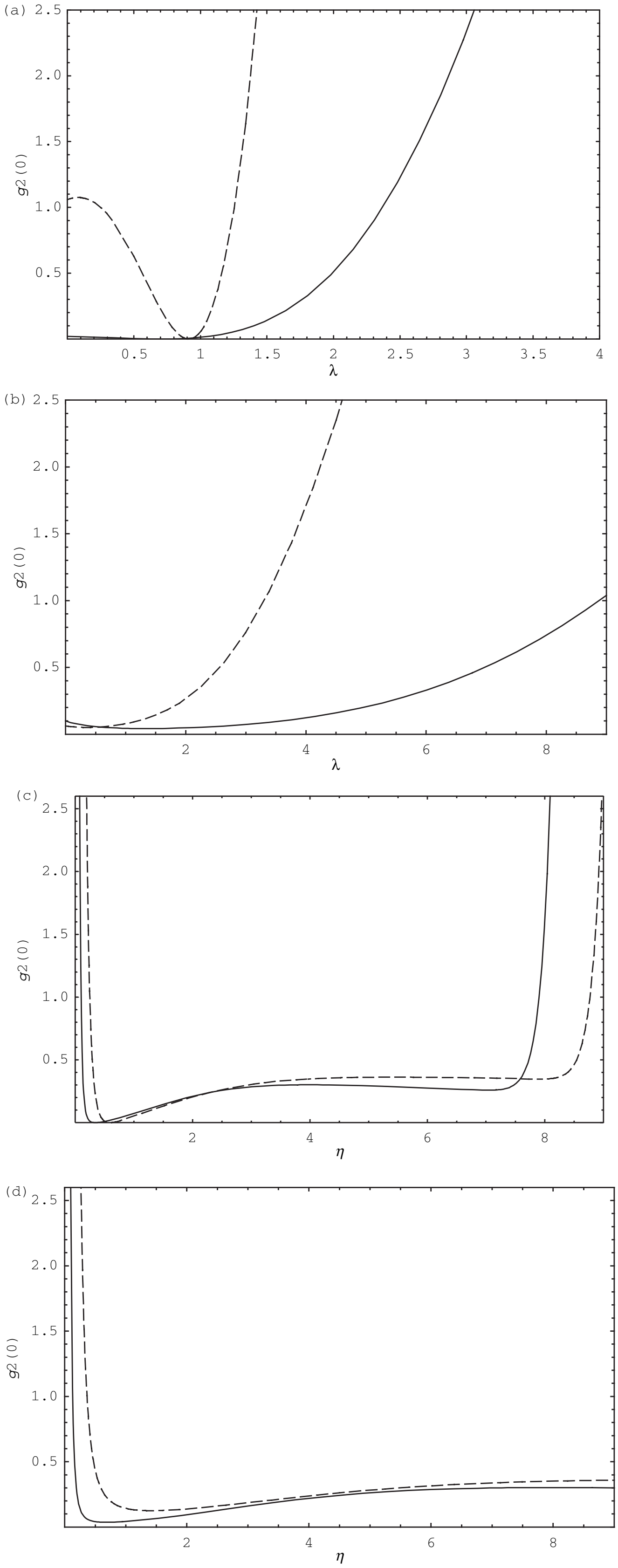

Figure 1. Coherence function $g^{(2)}$ measured on the vertical axis and the horizontal axis indicates the parameters λ in (a), (b) and η in (c), (d). The remaining parameters are: (a) $\eta = 1/4$, $k = 1/4$ (solid curve) and $\eta = 1/4$, $k = 3/4$ (dashed curve); (b) $\eta = 1$, $k = 1/4$ (solid curve) and $\eta = 1$, $k = 3/4$ (dashed curve). (c) $\lambda = 1/4$, $k = 1/4$ (solid curve) and $\lambda = 1/4$, $k = 3/4$ (dashed curve); (d) $\lambda = 1$, $k = 1/4$ (solid curve) and $\lambda = 1$, $k = 3/4$ (dashed curve).

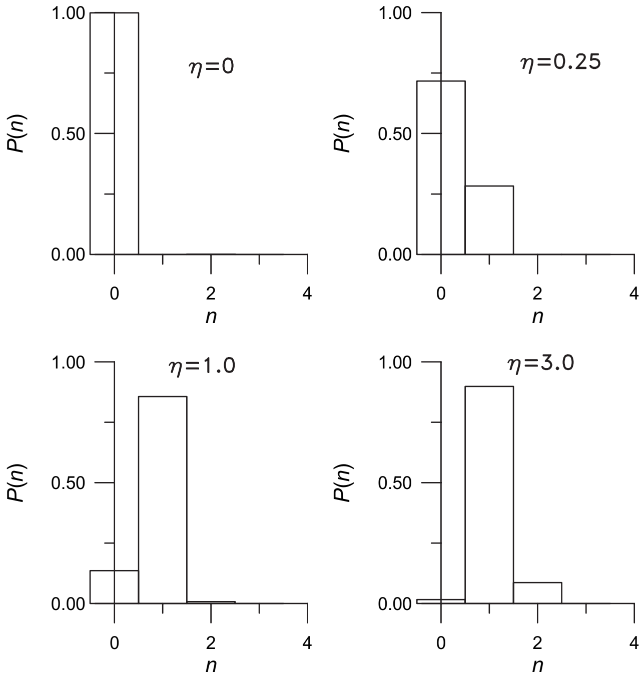

Figure 2. The photon number distribution $P(n)$ with $k = 1/4$ and $\lambda = 0.5$.

5.2. Particle number distribution

We start by looking at the particle number distribution for the state $|\Psi\rangle$. The number distribution $P_k(n)$ is given by

$$P_k(n) = |\langle k, n|\Psi\rangle|^2 = |c_n(k)|^2. \tag{5.6}$$

In figure 2, we illustrate the quantum number distribution $P_k(n)$ against η, for $\lambda = 0.5$ and $k = 14$. For $\eta = 0$, the distribution has its maximum at $n = 0$. The position of the mean peak of the number distribution $P(n)$ depends on the value of η. The function $P(n)$ does not show oscillations for the values of η and λ considered here. For $\eta = 0$, the vacuum is the most effective state. As η increases, other states start to be effective. Especially for $k = 1/4$, the state $n = 1$ becomes the most effective state for $\eta = 1.5$ with small contributions from other states. The photon-like number distribution introduced in this section is in Bergmann space with orthogonal basis, not the standard photon number distribution in the Fock space. Also, small oscillations of $P(n)$ appear for the NLCSs [62], in contrast to the results in [9] for the standard squeezed CSs which are depicted in [60].

5.3. Squeezing evolution for IS

The study of squeezing properties associated with the SU(1,1) states aims mainly at the reduction of the quantum noise in the act of measurement in the two important fields: spectroscopy [36, 75] and interferometry [76]. Squeezing fluctuations are important in quantum measurement and communication theories. In the SU(1,1) Lie group [36, 77], fluctuations in the quadrature operators K_1 and K_2 are squeezed if the following condition is satisfied:

$$(\Delta K_1)^2 < \tfrac{1}{2}|\langle K_3\rangle| \quad \text{or} \quad (\Delta K_2)^2 < \tfrac{1}{2}|\langle K_3\rangle|, \tag{5.7}$$

where the variances $(\Delta K_j)^2 = \langle K_j^2\rangle - \langle K_j\rangle^2$, $j = 1, 2$.

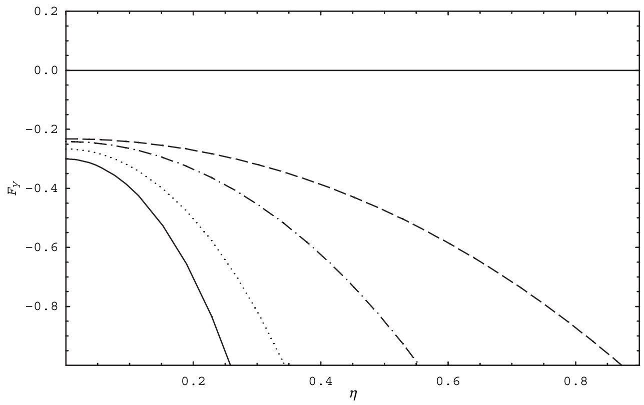

Figure 3. Squeezing parameter F_y against the parameter η with $k = 1/4$, and λ are: $\lambda = 0.1$ (solid curve), $\lambda = 0.5$ (dotted curve), $\lambda = 1.5$ (chained curve) and $\lambda = 3$ (dashed curve).

The condition of squeezing can be expressed as the reduction in the fluctuation of the K_1- or K_2-component through the quantities

$$F_x < 0 \quad \text{or} \quad F_y < 0, \tag{5.8}$$

respectively, where $F_{x,y} = (\Delta K_{1,2})^2 - \frac{1}{2}|\langle K_3\rangle|$, and the range of squeezing depends on the initial state.

In figure 3, F_y calculated from (3.12) is plotted against η, for various values of λ, while $k = 1/4$. From this figure, the SU(1,1) squeezing in the K_2-component is detected. As λ increases, it is noted that the amount of squeezing decreases for the same value of η. It is observed numerically that the Bergmann index affects the depth of squeezing in the K_2-component.

6. Discussion

In this paper, we have constructed and studied the SU(1,1) IS. Some known bosonic states are considered as special cases of these states. Especially, the bosonic systems (4.2) which have been studied as: squeezed states for general systems [78], squeezed CSs associated with the SU(1,1) [79] and second-order squeezed states [80]. Complete solutions for equation (4.2) have been introduced in the coordinate representation [78], in the CS-representation [79, 80] and in the number-state representation [80]. Statistical properties such as correlation functions and photon statistics have been examined, and can be compared with our results. In particular, the analytical solutions of equation (4.2) are given as formulae in terms of hypergeometric functions [80].

It is clear that SU(1,1) BGCSs and SU(1,1) PCSs are special classes of SU(1,1) IS. In the quantum information tools criteria, some studies have been carried out via SU(2) and SU(1,1) Lie algebra symmetries [81–89]. For quantum systems with semisimple Lie group symmetries, the only special classes of IS are pure states which factorize upon splitting [81, 86]; these states are the unique Bell states [81]. The limiting procedure for obtaining separable states (i.e. products of CSs) upon beam splitting by performing contractions of SU(2) and SU(1,1) Lie algebras and of their associated CSs has been examined [86]. In addition, the separability problem in group-theoretical terms is reformulated by changing the mathematical object representing quantum states [88]. A method of generating the entangled SU(1,1) CSs of two types of PCSs and BGCSs is discussed [82]. The entanglement of SU(1,1) CSs with $k = 1/4$ (single-mode squeezed vacuum state) and with $k = 3/4$ (single-mode squeezed one-photon state) is considered [82]. Several aspects of quantum information criteria, such as entanglement, degree of entanglement, qubit, Bell states and entropy for SU(1,1) CSs are explored in [82]. In an arbitrary SU(2) transformation, qubits can be encoded in program state of a universal programmable probabilistic quantum processor as reported in [85].

In the two-mode bosonic realization of SU(1,1) Lie algebra, the corresponding PCS is a maximum entangled state [83]. For the nonlinear realization of SU(1,1) IS, we expect it to define the Bell states which can be generated from a physical system [89]. Also, in a bipartite composite system, a normalizable BGCS for quantized q-deformed SU(1,1) Lie algebra has been constructed and the entanglements have been investigated. Finally, we remark that the general nonlinear, q-deformed, or nonlinear realization of SU(1,1) Lie algebra may be used for obtaining entangled states of bipartite composite systems.

Some investigations have been devoted to testing the entanglement in non-Gaussian states of discrete and continuous variables [88, 90–92]. The separability conditions that can be obtained from the uncertainty relations in the SU(2) and SU(1,1) Lie algebra have been studied in some detail in [88]. The class of inequalities obtained in [88] by using the minimum uncertainty relation and applying it to the two-mode IS of SU(2), has been included as special cases of the work done in [90, 91]. This class is optimal in detecting entanglement for a broad class of non-Gaussian entangled states. These recent measures and conditions of entanglement have been applied to the study of several linear devices, the beam splitter, the parametric amplifier and the linear phase-insensitive amplifier. It is important to study in detail the SU(1,1) IS with quantum information criteria, and these ideas may be considered as topics for future research.

The SU(1,1) Lie algebra is used in studying the adaptation of Kieu's hypercomputational quantum algorithm [93] and for quantum computation and tomography [94]. The SU(1,1) dynamical algebra is selected, because it possesses the necessary characteristics in realizing the hypercomputational quantum algorithm. In addition, it admits different kinds of CSs and various kinds of representations. Some realizations of SU(1,1) Lie algebra such as Holstein–Primakoff and one-mode nonlinear realization are used for studying the quantum hypercomputation [93]. Moreover, the approximation correspondence between the SU(1,1) Lie algebra and finite elements of quantum gates are derived [94].

7. Conclusions

In this work, we have studied the SU(1,1) Lie algebra IS. The generalized CSs associated with the SU(1,1) Lie algebra have been reviewed. The eigenvalue problem satisfied by the IS has been solved. The solution is related to the Pollaczek polynomials. Some realizations for our results of

SU(1,1) group IS have been investigated. The IS are a wider class of other SU(1,1) group states. Some bosonic states are considered as special cases of these states.

We have discussed numerically some properties of these states. Several moments have been calculated. Some nonclassical properties such as sub-Poissonian behaviour, particle number distribution and squeezing are investigated. The correlation function $g^{(2)}$ has been investigated numerically and it indicates that the IS exhibit sub-Poissonian behaviour. Depending on the parameters of these states large areas of squeezing appear, which are considered as a measure of nonclassicality.

Acknowledgment

The authors would like to thank Professor Roger Wappling for carefully reading the manuscript which helped to improve the matter of this paper.

References

[1] Weyl H 1927 *Z. Phys.* **46** 1
Weyl H 1950 *The Theory of Groups and Quantum Mechanics* (New York: Dover)
[2] Jordan T F and Sudarshan E C G 1961 *Rev. Mod. Phys.* **33** 515
[3] Arvind P K, Dutta B, Mukunda N and Simon R 1995 *Pramana J. Phys.* **45** 471
[4] Wünsche A 2000 *J. Opt B: Quantum Semiclass. Opt.* **2** 73
[5] Glauber R J 1963 *Phys. Rev.* **131** 2766
[6] Stoler D 1970 *Phys. Rev.* D **1** 3217
Stoler D 1971 *Phys. Rev.* D **4** 1925
[7] Yuen H P 1976 *Phys. Rev.* A **13** 2226
[8] Walls D F 1983 *Nature* **306** 141
[9] Loudon R and Knight P L 1987 *J. Mod. Opt.* **34** 709
Schleich W and Wheeler J A 1987 *J. Opt. Soc. Am.* B **4** 1715
[10] Hirota O 1992 *Squeezed Light* (Amsterdam: Elsevier)
[11] Dodonov V V 2002 *J. Opt. B: Quantum Semiclass. Opt.* **4** R1–33
[12] Klauder J R 1963 *J. Math. Phys.* **4** 1055–8
[13] Klauder J R and Skagerstam B S 1985 *Coherent States Applications in Physics and Mathematical Physics* (Singapore: World Scientific)
[14] Zhang W-M, Feng D H and Gilmore R 1990 *Rev. Mod. Phys.* **62** 867
Kastrup H A 2007 *Ann. Phys., Lpz.* **16** 439–528
[15] Barut A O and Girardello L 1971 *Commun. Math. Phys.* **21** 41
[16] Perelomov A M 1972 *Commun. Math. Phys.* **26** 222
[17] Perelomov A M 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[18] Wodkiewicz K and Eberly J H 1985 *J. Opt. Soc. Am.* B **2** 458
[19] Basu D 1992 *J. Math. Phys.* **33** 114
[20] Wünsche A 2003 *J. Opt. B: Quantum Semiclass. Opt.* **5** s429
[21] Gerry C C 2001 *Opt. Express* **8** 76
[22] Novaes M 2004 *Rev. Brasil. Ensino Fis.* **26** 351
[23] Aragone C, Guerri G, Salamo S and Tani J L 1974 *J. Phys. A: Math. Gen.* **7** L149
[24] Aragone C, Chalbaud E and Salamo S 1976 *J. Math. Phys.* **17** 1963
[25] Vandeb-Berghe G and DeMeyer H 1978 *J. Phys. A: Math. Gen.* **11** 1569
[26] Rashid M A 1978 *J. Math. Phys.* **19** 1391
[27] Rashid M A 1978 *J. Math. Phys.* **19** 1397
[28] Puri R R and Agarwal G S 1996 *Phys. Rev.* A **53** 1786
[29] Fu H-C and Sasaki R 1996 *Phys. Rev.* A **53** 3836
[30] Brif C 1997 *Int. J. Theor. Phys.* **36** 1651
[31] Brif C 1996 *Ann. Phys.* **251** 180
[32] Gerry C C and Grobe R 1995 *Phys. Rev.* A **51** 4123
[33] Brif C and Mann A 1996 *Phys. Rev.* A **54** 4505
Brif C and Mann A 1997 *Quantum Semiclass. Opt.* **9** 889
[34] Wineland D J, Monroe C, Itano W M, Leibfried D, King B E and Meekhof D M 1998 *J. Res. Nat. Inst. Stand. Technol.* **103** 259
[35] Gerry C C, Gou S C and Steinbach J 1997 *Phys. Rev.* A **55** 630
[36] Yurke B, McCall S L and Klauder J R 1986 *Phys. Rev.* A **33** 4033
Perinova V, Lukš A and Krepelka J 2000 *J. Opt. B: Quantum Semiclass. Opt.* **2** 81
[37] Brif C and Ben-Aryeh Y 1994 *J. Phys. A: Math. Gen.* **27** 8185
[38] Gerry C C and Grobe R 1997 *Quantum Semiclass. Opt.* **9** 59
[39] Milks M M and de Guise H 2005 *J. Opt B: Quantum Semiclass. Opt.* **7** s622
[40] Götte J B, Zambrini R, Franke-Arnold S and Barnett S M 2005 *J. Opt B: Quantum Semiclass. Opt.* **7** s563
Götte J B, Radmore P M, Zambrini R and Barnett S M 2006 *J. Phys. B: At. Mol. Opt. Phys.* **39** 2791
[41] Trifonov D A 1994 *J. Math. Phys.* **35** 2297
Aleixo A N F and Balantckin A B 2007 *J. Phys. A: Math. Theor.* **40** 5105
[42] El Kinani A H and Daud M 2001 *Int. J. Mod. Phys.* B **15** 2465
[43] Daud M 2002 *Mod. Phys. Lett.* A **17** 1805
[44] El Kinani A H and Daud M 2002 *Int. J. Mod. Phys.* B **16** 3915
[45] El Kinani A H and Daud M 2002 *J. Math. Phys.* **43** 714
[46] Roy B and Roychoudhury R 2003 *Int. J. Mod. Phys.* **17** 4631
[47] Daud M 2004 *Phys. Lett.* A **329** 318
[48] Daud M 2004 *J. Math. Phys.* **45** 3435
[49] Erdélyi A 1953 *Higher Transcendental Functions* vol 1 (New York: McGraw-Hill) p 103, 2.8(31)
Erdélyi A 1953 *Higher Transcendental Functions* vol 2 (New York: McGraw-Hill) p 220
[50] Nagel B 1995 *Spectra and Generalized Eigenfunctions of the One- and Two-Mode Squeezing Operators in Quantum Optics in Modern Group Theoretical Methods in Physics,* ed J Bertrand *et al* (Dordrecht: Kluwer Academic)
Nagel B 1997 *Preprint* quant-ph/9711018 p 211
[51] Hillery M 1987 *Phys. Rev.* A **36** 3796
[52] Bergou J A, Hillery M and Yu D 1991 *Phys. Rev.* A **43** 515
[53] Yu D and Hillery M 1994 *Quantum Opt.* **6** 37
[54] Nieto M M and Truax D R 2000 *Opt. Commun.* **179** 197
[55] Yurke B and Stoler D 1986 *Phys. Rev. Lett.* **57** 13
Yurke B and Stoler D 1988 *Physica* B **151** 298
[56] Dodonov V V, Malkin I A and Man'ko V I 1974 *Physica* **72** 597
Buzek V, Vidiella-Barranco A and Knight P L 1992 *Phys. Rev.* A **45** 6570
[57] de Matos Filho R L and Vogel W 1996 *Phys. Rev.* A **54** 4560
[58] Manko V I, Marmo G, Sudarshan E C G and Zaccaria F 1997 *Phys. Scr.* **55** 528
Manko V I, Marmo G, Porzio A, Solimeno S and Zaccaria F 2000 *Phys. Rev.* A **62** 053407
[59] Roy B 1998 *Phys. Lett.* A **249** 25
[60] Obada A-S F and Abd Al-Kader G M 2005 *J. Opt B: Quantum Semiclass. Opt.* **7** s635
[61] Obada A-S F and Darwish M 2005 *J. Opt. B: Quantum Semiclass. Opt.* **7** 57
[62] Obada A-S F and Abd Al-Kader G M 2007 *Eur. Phys. J.* D **41** 189
[63] Vourdas A 1992 *Phys. Rev.* A **46** 442
[64] Bhaumik D, Bhaumik K and Dutta-Roy B 1976 *J. Phys. A: Math. Gen.* **9** 1507
Agarwal G S 1988 *J. Opt. Soc. Am.* B **5** 1940
[65] Gerry C C 1991 *J. Opt. Soc. Am.* B **8** 685
Gerry C C and Benmoussa A 2000 *Phys. Rev.* A **62** 033812
[66] Obada A-S F, Khalil E M and Abd Al-Kader G M 2005 *Al-Azhar Univ. Eng. J.* **8** 83
Obada A-S F, Abdel-Aty M, Khalil E M and Abd Al-Kader G M 2007 *Phys. Scr.* **75** 557
Khalil E M and Abd Al-Kader G M 2007 *Aspects in Optical Science and Applied Mathematics* (a special review book in the Physical Sciences section of Research Signpost 2007)
[67] Wang X G 2000 *Opt. Commun.* **178** 365
[68] Davidovich L 1996 *Rev. Mod. Phys.* **68** 127

[69] Titulaer U M and Glauber R J 1965 *Phys. Rev.* **140** B676
Wünsche A, Dodonov V V, Man'ko O V and Man'ko V I 2001 *Fortschr. Phys.* **49** 1117
[70] Richter Th and Vogel W 2002 *Phys. Rev. Lett.* **89** 283601
[71] Shchukin E V and Vogel W 2005 *Phys. Rev.* A **72** 043808
[72] Shchukin E V and Vogel W 2006 *Phys. Rev. Lett.* **96** 200403
[73] Kenfack A and Życzkowski K 2004 *J. Opt. B: Quantum Semiclass. Opt.* **6** 396
[74] Ban M 1992 *J. Math. Phys.* **33** 3213
Ban M 1993 *J. Opt. Soc. Am.* B **10** 1347
Ban M 1994 *Phys. Lett.* A **193** 121
[75] Caves C M 1981 *Phys. Rev.* D **23** 1693
Bondurant R S and Shapiro J H 1984 *Phys. Rev.* D **30** 2584
[76] Allen L and Eberly J H 1975 *Optical Resonance and 2-level Atoms* (New York: Wiley)
[77] Buzek V 1990 *J. Mod. Opt.* **37** 303
[78] Nieto M M and Truax D R 1993 *Phys. Rev. Lett.* **71** 2843
[79] Prakash G S and Agarwal G S 1994 *Phys. Rev.* A **50** 4258
[80] Marian P 1997 *Phys. Rev.* A **55** 3051
Marian P 1999 *Phys. Rev.* A **59** 3141
[81] Brif C, Mann A and Revzen M 1998 *Phys. Rev.* A **57** 742
[82] Wang X-G, Sanders B C and Pan S 2000 *J. Phys. A: Math. Gen.* **33** 7451
Shaterzadeh-Yazdi Z, Turner P S and Sanders B C 2008 *J. Phys. A: Math. Theor.* **41** 055309
[83] Santana A E, Khanna F C and Revzen M 2002 *Phys. Rev.* A **65** 032119
[84] Chakrabarti R and Vasan S S 2003 *Phys. Lett.* A **312** 287
[85] Bužek V, Ziman M and Hillery M 2004 *Fortschr. Phys.* **52** 1056
[86] Gerry C C and Benmoussa A 2005 *Phys. Rev.* A **71** 062319
[87] Nha H and Kim J 2006 *Phys. Rev.* A **74** 012317
[88] Korbicz J K and Lewenstein M 2006 *Preprint* quant-ph/0601189
[89] Khalil E M 2006 *J. Phys. A: Math. Gen.* **39** 11053
[90] Hillery M and Zubairy M S 2006 *Phys. Rev. Lett.* **96** 050503
[91] Agarwal G S and Biswas A 2005 *New J. Phys.* **7** 211
[92] Hillery M and Zubairy M S 2006 *Preprint* quant-ph/0606154
[93] Sicrd A, Ospina J and Velez M 2006 *J. Phys. A: Math. Gen.* **39** 12539
[94] Chiribella G, D'Ariano M and Perinotti P 2006 *Laser Phys.* **16** 1572