The intelligent states. I. Group-theoretic study and the computation of matrix elements

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In this first of a series of papers, a group-theoretic study is presented of the quasi-intelligent states which are a generalization of the intelligent states satisfying equality in the Heisenberg uncertainty relation $\Delta J_1^2 \Delta J_2^2 \geq (1/4)|\langle J_3\rangle|^2$. A method based on the knowledge of a certain generating function is given for the calculation of matrix elements of polynomials in the infinitesimal generators of the rotation group between quasi-intelligent states. Examples of such computations are also included to exhibit the improvement and efficiency of the present methods.

I. INTRODUCTION

Aragone *et al.*[1] have recently considered the states of a well-defined angular momentum which satisfy equality $\Delta J_1^2 \Delta J_2^2 = \frac{1}{4}|\langle J_3\rangle|^2$ in the Heisenberg uncertainty relation derived from the commutation relation $[J_1, J_2] = iJ_3$. Unlike the Glauber coherent states of a linear harmonic oscillator, these states are not generally the minimum uncertainty states, i.e., $\Delta J_1^2 \Delta J_2^2$ does not take a minimum value for them.

In the following series of papers, we shall attempt to present a somewhat different but more manifest method of handling these states which are called intelligent states in the literature. Our papers will clarify the algebraic structure of these states and emphasize the distinction between them and the usual Wigner states $|jm\rangle$.

In the present paper, we introduce the group-theoretic structure and present methods for the computation of elementary matrix elements of the generators between these states. In the second paper of the series we examine the problem of the computation of the Clebsch–Gordan coefficients for the intelligent states. In the third paper, we hope to present certain physical applications.

The present paper is organized as follows. In Sec. II, we repeat briefly, for completeness, the argument that the states which satisfy equality in the Heisenberg uncertainty relation are indeed eigenstates of a *non-Hermitian operator* J_3' with a known spectrum. In Sec. III, we give the operators which together with J_3' form the *same* algebra as that formed by the infinitesimal generators of the three-dimensional rotation group. In this section, we also present a compact representation of these states up to normalization in terms of the operation of the infinitesimal generators of the rotation group on the Wigner states. A simple expression for the normalization coefficients is also obtained in this section.

In Sec. IV, we arrive at a manifest connection between the intelligent states and the Wigner states. This connection also leads to another, somewhat more complicated, expression for the normalization coefficients which is shown to be equivalent to the simpler expression presented in Sec. III.

Section V is devoted to the computation of some elementary matrix elements by mentioning that a certain generating function is trivially calculable using our methods. With the use of this generating function, these elementary matrix elements can easily be calculated. This section ends with a few examples to illustrate the efficiency of our approach.

II. THE HEISENBERG UNCERTAINTY RELATION

Let us start with the commutator $[A, B] = iC$, where A and B are Hermitian (and hence C is also Hermitian). For any state ψ, defining

$$(\Delta A)^2 = \int \psi^*(A - \langle A\rangle)^2 \psi \, d\tau, \tag{1}$$

we obtain

$$(\Delta A)^2 = \int |(A - \langle A\rangle)|^2 \, d\tau \tag{2}$$

since A is Hermitian.

Now we use the Schwartz inequality

$$\int |f|^2 d\tau \int |g|^2 d\tau \geq \int |f^* g|^2 d\tau \geq \left|\int f^* g \, d\tau\right|, \tag{3}$$

with $f = (A - \langle A\rangle)\psi$ and $g = (B - \langle B\rangle)\psi$. This results in

$$(\Delta A)^2 (\Delta B)^2 \geq \left|\int \psi^*(A - \langle A\rangle)(B - \langle B\rangle)\psi \, d\tau\right|^2, \tag{4}$$

where the *equality* sign will hold if and only if

$$(B - \langle B\rangle)\psi = \lambda(A - \langle A\rangle)\psi, \tag{5}$$

where, so far, λ is any (possibly a complex) number.

Next we try to relate the right-hand side in the inequality given in Eq. (4) above to $\langle C\rangle$. We note that

$$\begin{aligned}(A - \langle A\rangle)&(B - \langle B\rangle)\\ &= \tfrac{1}{2}[(A - \langle A\rangle)(B - \langle B\rangle) + (B - \langle B\rangle)(A - \langle A\rangle)]\\ &\quad + \tfrac{1}{2}[(A - \langle A\rangle)(B - \langle B\rangle) - (B - \langle B\rangle)(A - \langle A\rangle)]\\ &= F + \tfrac{1}{2}iC,\end{aligned} \tag{6}$$

where

$$F = \tfrac{1}{2}[(A - \langle A\rangle)(B - \langle B\rangle) + (B - \langle B\rangle)(A - \langle A\rangle)] \tag{7a}$$

and

$$(A - \langle A\rangle)(B - \langle B\rangle) - (B - \langle B\rangle)(A - \langle A\rangle) = [A, B] = iC. \tag{7b}$$

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Since F and C are Hermitian operators, $\langle F\rangle$ and $\langle C\rangle$ are both real numbers and we find

$$|\langle F+\tfrac{1}{2}iC\rangle|^2=|\langle F\rangle|^2+\tfrac{1}{4}|\langle C\rangle|^2\geq\tfrac{1}{4}|\langle C\rangle|^2, \tag{8}$$

where again the equality will hold provided $\langle F\rangle=0$.

Combining Eqs. (4), (6), and (8), we arrive at

$$(\Delta A)^2(\Delta B)^2\geq\tfrac{1}{4}|\langle C\rangle|^2, \tag{9}$$

which is the well-known Heisenberg uncertainty relation.[2]

Our interest is basically in understanding when we shall have an equality in Eq. (9). From the argument presented above, it is now clear that the equality will hold for those states ψ for which

$$(B-\langle B\rangle)\psi=\lambda(A-\langle A\rangle)\psi \tag{10a}$$

and

$$\langle F\rangle=0. \tag{10b}$$

Equations (7) and (10) now lead to

$$\lambda(\Delta A)^2+\frac{1}{\lambda}(\Delta B)^2=0 \tag{11a}$$

and

$$\lambda(\Delta A)^2-\frac{1}{\lambda}(\Delta B)^2=i\langle C\rangle, \tag{11b}$$

which imply

$$\lambda=\tfrac{1}{2}i\,\frac{\langle C\rangle}{(\Delta A)^2}, \tag{12}$$

where since $\langle C\rangle$ and $(\Delta A)^2$ are both real, λ is indeed pure imaginary. This shows that the states ψ for which the Heisenberg uncertainty relation has an equality are those for which

$$(A-i\alpha B)\psi=(\langle A\rangle-i\alpha\langle B\rangle)\psi, \tag{13}$$

i.e., they are eigenstates of the operator $A-i\alpha B$ for *real* α. (Note that we have replaced the purely imaginary number λ^{-1} by $i\alpha$.)

Let us now apply the above result to the special case where A, B, C are J_1, J_2, J_3—the generators of the infinitesimal rotations in the three-dimensional space. Then we note that the states for which

$$(\Delta J_1)^2(\Delta J_2)^2=\tfrac{1}{4}|\langle J_3\rangle|^2$$

are eigenstates of the non-Hermitian operator $J_1-i\alpha J_2$ for some real α. In the following, we shall explicitly determine these states (called the intelligent states in the literature) for a given angular momentum j as a linear combination of the Wigner states $|jm\rangle$ and also study their properties.

III. THE OPERATORS $J_3'(\alpha)$ AND $J_\pm'(\alpha)$

Though intelligent states correspond to real α only, we shall consider the more general situation, where α is any complex number (the corresponding eigenstates of $J_1-i\alpha J_2$ may be called quasi-intelligent states).

We define[3]

$$J_3'(\alpha)=\frac{J_1-i\alpha J_2}{(1-\alpha^2)^{1/2}} \tag{14a}$$

and

$$J_\pm'(\alpha)=\mp\frac{\alpha}{(1-\alpha^2)^{1/2}}J_1\pm\frac{i}{(1-\alpha^2)^{1/2}}J_2-J_3 \tag{14b}$$

for any complex $\alpha\neq\pm1$. (This restriction will be clear soon.) The operators $J_3'(\alpha)$ and $J_\pm'(\alpha)$ satisfy the commutation relations

$$[J_3'(\alpha),J_\pm'(\alpha)]=\pm J_\pm'(\alpha) \tag{15a}$$

and

$$[J_+'(\alpha),J_-'(\alpha)]=2J_3'(\alpha), \tag{15b}$$

which are exactly the same as those satisfied by $J_3, J_\pm=J_1\pm iJ_2$. Also

$$\begin{aligned}\mathbf{J}^2&=J_1^2+J_2^2+J_3^2=\tfrac{1}{2}(J_+J_-+J_-J_+)+J_3^2\\&=\tfrac{1}{2}[J_+'(\alpha)J_-'(\alpha)+J_-'(\alpha)J_+'(\alpha)]+J_3'^2(\alpha).\end{aligned} \tag{16}$$

We try to construct the eigenstates of the operator $J_3'(\alpha)$ as a linear combination of the states $|jm\rangle$ with a given j and $-j\leq m\leq j$, where these $(2j+1)$ Wigner states are the eigenstates of the *Hermitian operator* J_3. Indeed

$$J_3|jm\rangle=m|jm\rangle \tag{17a}$$

and

$$J_\pm|jm\rangle=\sqrt{(j\mp m)(j\pm m+1)}\,|jm\rangle. \tag{17b}$$

Since J_3 and $\mathbf{J}^2$ are *Hermitian operators*, the states $|jm\rangle$ can be orthonormalized in the form

$$\langle j'm'|jm\rangle=\delta_{jj'}\delta_{mm'}, \tag{18}$$

which is what one conventionally does.

Noting that

$$\exp(J_3\theta)J_1\exp(-J_3\theta)=J_1\cosh\theta+iJ_2\sinh\theta, \tag{19}$$

we realize that the right-hand side will be proportional to $J_1-i\alpha J_2$ provided one chooses θ such that

$$\cosh\theta=\frac{1}{(1-\alpha^2)^{1/2}}\quad\text{and}\quad\sinh\theta=-\frac{\alpha}{(1-\alpha^2)^{1/2}} \tag{20}$$

or

$$e^\theta=\left(\frac{1-\alpha}{1+\alpha}\right)^{1/2}=\tau \tag{21}$$

and then the right-hand side of Eq. (19) is just $J_3'(\alpha)$. With the above choice of θ, we find

$$\exp(\pm J_3\theta)J_1\exp(\mp J_3\theta)=\frac{J_1\mp i\alpha J_2}{(1-\alpha^2)^{1/2}} \tag{22a}$$

and

$$\exp(\pm J_3\theta)J_2\exp(\mp J_3\theta)=\frac{\pm i\alpha J_1+J_2}{(1-\alpha^2)^{1/2}}. \tag{22b}$$

In particular,

$$\exp(-J_3\theta)J_3'(\alpha)\exp(J_3\theta)=J_1. \tag{23}$$

Noting also that

$$\exp(\pm i\tfrac{1}{2}\pi J_2)J_1\exp(\mp i\tfrac{1}{2}\pi J_2)=\pm J_3, \tag{24}$$

we immediately see that the state

$$|jm\alpha\rangle'=\exp(\theta J_3)\exp(-i\tfrac{1}{2}\pi J_2)|jm\rangle \tag{25}$$

is indeed an eigenstate of the operator $J_3'(\alpha)$ with the eigenvalue m. The prime on $|jm\alpha\rangle$ is indicative of the fact that the state as defined may not be normalized.

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Since θ is not necessarily pure imaginary (this requires $|\tau|=1$ or α = pure imaginary—for intelligent states θ is definitely not pure imaginary[4]) the above state does not, in general, correspond to a rotation of the Wigner state $|jm\rangle$. In fact, as we shall see, the above state is not normalized and the $(2j+1)$ states $|jm\alpha\rangle'$ for $-j\leq m\leq j$ are not orthogonal unless θ is purely imaginary. *The basic reason for this is the non-Hermiticity of the operator $J_3'(\alpha)$ of which these are eigenstates.*

Next we attempt to compute the overlap

$$
\begin{aligned}
&{}'\langle jm'\alpha'|jm\alpha\rangle' \\
&=\langle jm'|\exp(i\tfrac{1}{2}\pi J_2)\exp(\theta'^*J_3)\exp(\theta J_3)\exp(-i\tfrac{1}{2}\pi J_2)|jm\rangle \\
&=\langle jm'|\exp[-(\theta+\theta'^*)J_1]|jm\rangle .
\end{aligned} \tag{26}
$$

The above matrix element can immediately be computed using the 2×2 representation

$$J_1\cong\tfrac{1}{2}\sigma_1=\tfrac{1}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

of J_1, in terms of which

$$
\exp[-(\theta+\theta'^*)J_1] \cong \begin{bmatrix}\cosh\left(\dfrac{\theta+\theta'^*}{2}\right) & -\sinh\left(\dfrac{\theta+\theta'^*}{2}\right)\\ -\sinh\left(\dfrac{\theta+\theta'^*}{2}\right) & \cosh\left(\dfrac{\theta+\theta'^*}{2}\right)\end{bmatrix} \tag{27}
$$

Thus we find

$$
\begin{aligned}
&{}'\langle jm'\alpha'|jm\alpha\rangle' \\
&=\langle jm'|\exp[-(\theta+\theta'^*)J_1]|jm\rangle \\
&=[(j+m)!(j-m)!(j+m')!(j-m')!]^{1/2}(-1)^{2j+m+m'} \\
&\quad\times\sum_r\frac{[\cosh((\theta+\theta'^*)/2)]^{-m-m'+2r}[\sinh((\theta+\theta'^*)/2)]^{2j+m+m'-2r}}{r!(j+m-r)!(j+m'-r)!(-m-m'+r)!}
\end{aligned} \tag{28a}
$$

$$
\begin{aligned}
&=\left(\frac{(j+m)!(j-m')!}{(j-m)!(j+m')!}\right)^{1/2}\left[\sinh\left(\frac{\theta+\theta'^*}{2}\right)\right]^{m'-m}(-1)^{2j+m+m'} \\
&\quad\times\sum_r(-1)^r\frac{(2j-r)![\cosh((\theta+\theta'^*)/2)]^{2j+m-m'-2r}}{r!(j+m-r)!(j-m'-r)!} .
\end{aligned} \tag{28b}
$$

In particular, if we define the normalized state by

$$
\begin{aligned}
|jm\alpha\rangle&=(a^j{}_m(\alpha))^{-1}|jm\alpha\rangle' \\
&=(a^j{}_m(\alpha))^{-1}\exp(\theta J_3)\exp(-i\tfrac{1}{2}\pi J_2)|jm\rangle,
\end{aligned} \tag{29}
$$

then the normalization coefficient $a^j{}_m(\alpha)$ is given by

$$
a^j{}_m(\alpha)=\left[(j+m)!(j-m)!\sum_r\frac{[\cosh((\theta+\theta^*)/2)]^{-2m+2r}}{r!(j+m-r)!}\times\frac{[\sinh((\theta+\theta^*)/2)]^{2j+2m-2r}}{(j+m-r)!(-2m+r)!}\right]^{1/2} . \tag{30a}
$$

$$
=\left[\sum_r(-1)^r\frac{(2j-r)![\cosh((\theta+\theta^*)/2)]^{2j-2r}}{r!(j+m-r)!(j-m-r)!}\right]^{1/2} . \tag{30b}
$$

In Eq. (30a), since $\sinh((\theta+\theta^*)/2)$ is raised to an *even* power, every term in the sum is nonnegative $(\theta+\theta^*$ is a real quantity) and hence the square root is well defined.

Since the angle θ in everything we consider appears as a function of $\exp(\theta/2)=\tau^{1/2}$, we shall choose the τ complex plane to be cut from $-\infty$ to 0. The relationship between α and τ now is

$$\tau=\left(\frac{1-\alpha}{1+\alpha}\right)^{1/2}\quad\text{or}\quad \alpha=\frac{1-\tau^2}{1+\tau^2}. \tag{31a}$$

In particular

$$1-\alpha^2=\frac{4\tau^2}{(1+\tau^2)^2} \tag{31b}$$

and we choose

$$(1-\alpha^2)^{1/2}=\frac{2\tau}{1+\tau^2}. \tag{31c}$$

Thus any expression in terms of θ and α can immediately be expressed in terms of the variable τ. For real α (i.e., when we are considering intelligent states only), τ will be taken as positive real or positive imaginary. Indeed, all the α plane is obtained from the upper half-plane of τ.

We end this section with a few remarks.

(1) The matrix elements $\langle jm'|\exp(-\theta J_1)|jm\rangle$ are evidently [see Eq. (28a) with θ in place of $\theta+\theta'^*$] symmetrical in the interchange $m\leftrightarrow m'$. For real θ, these are also real. The expression in Eq. (28b) is not manifestly symmetrical under this interchange, but it is presented on account of its simplicity and usefulness.

(2) The matrix elements $\langle jm'|\exp(-\theta J_1)|jm\rangle$ *for real* θ cannot be zero except when $\theta=0$ since every term on the right-hand side in Eq. (28a) is positive for $\theta<0$ and is positive (negative) for $\theta>0$ whenever $2j+m+m'$ is even (odd). This shows that the states $|jm\alpha\rangle$, $-j\leq m\leq j$, for given j and α are not orthogonal unless real $\theta=0$. The quasi-intelligent states for a given j and α are thus necessarily nonorthogonal unless $|\tau|=1$ or α pure imaginary.

(3) The normalization factors $a^j{}_m(\alpha)$ for a given j and α are only 1 when real $\theta=\frac{1}{2}(\theta+\theta^*)=0$. Indeed whenever real $\theta=0$, $[a^j{}_m(\alpha)]^2=1$ as is obvious from Eq. (30a). For real $\theta\neq 0$, we can differentiate the expression for $[a^j{}_m(\alpha)]^2$ obtained from Eq. (30a) and note that the derivative has the sign of $\tanh((\theta+\theta^*)/2)$. Thus the norms $a^j{}_m(\alpha)\gtrless 1$ for $\theta+\theta^*\gtrless 0$.

Aragone *et al.*[5] could not find the properties mentioned in remarks (2) and (3) above, since they lacked simple analytic expressions for the matrix elements of the form $\langle jm'|\exp(-\theta J_1)|jm\rangle$.

(4) In the next section, we shall rewrite Eq. (29) expressing the states $|jm\alpha\rangle$ for any given m as a linear combination of the $(2j+1)$ Wigner states $|jm'\rangle$, $-j\leq m'\leq j$. This process can also be inverted, i.e., we can express any Wigner state $|jm\rangle$ in terms of the $(2j+1)$ quasi-intelligent states $|jm'\alpha\rangle$, $-j\leq m'\leq j$ for a given $\alpha\neq\pm 1$. (This inversion will be presented in the second paper.) In this sense, therefore, the $(2j+1)$ quasi-intelligent states $|jm'\alpha\rangle$, $-j\leq m'\leq j$ for a given $\alpha\neq\pm 1$ are complete. For $\alpha=+1(-1)$, $J_3'(\alpha)=J_1-iJ_2$

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(J_1+iJ_2) is the usual lowering (raising) operator for the Wigner states. The only eigenstate it has is the Wigner state $|j,-j\rangle$ ($|j,j\rangle$) with the eigenvalue 0. Thus for $\alpha=\pm 1$, the analysis presented above completely breaks down. The question of completeness for $|jm\alpha\rangle$, $\alpha=\pm 1$ is just not there. This is the reason why in the beginning of the present section we restricted ourselves to $\alpha\neq\pm 1$.

IV. RELATIONSHIP BETWEEN THE QUASI-INTELLIGENT STATES AND THE WIGNER STATES

We have seen that the normalized quasi-intelligent states for given j,α are given by

$$|jm\alpha\rangle=(a^j{}_m(\alpha))^{-1}\exp(\theta J_3)\exp(-i\tfrac{1}{2}\pi J_2)|jm\rangle. \tag{29'}$$

Again we can use the 2×2 representation $J_i\cong\frac{1}{2}\sigma_i$ for the operators J_i [6] and employ the standard techniques to arrive at

$$|jm\alpha\rangle=(a^j{}_m(\alpha))^{-1}2^{-j}\sum_{m'r}|jm'\rangle\exp(m'\theta)(-1)^{j+m'-r}$$
$$\times\frac{[(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!]^{1/2}}{r!(j+m-r)!\,(j+m'-r)!\,(-m-m'+r)!} \tag{32a}$$
$$=(a^j{}_m(\alpha))^{-1}2^{-j}\sum_{m'r}|jm'\rangle\exp(m'\theta)\,2^r(-1)^r$$
$$\times\left[\frac{(j-m)!\,(j+m')!}{(j+m)!\,(j-m')!}\right]^{1/2}\frac{(2j-r)!}{r!\,(j-m-r)!\,(j+m'-r)!} \tag{32b}$$
$$=(a^j{}_m(\alpha))^{-1}2^{-j}\sum_{m'r}|jm'\rangle\exp(m'\theta)\,2^r(-1)^{m'-m+r}$$
$$\times\left[\frac{(j+m)!\,(j-m')!}{(j-m)!\,(j+m')!}\right]^{1/2}\frac{(2j-r)!}{r!\,(j+m-r)!\,(j-m'-r)!} \tag{32c}$$

Using the above results in various combinations, we obtain several equivalent expressions for the inner product $\langle jm'\alpha'|jm\alpha\rangle$. Thus, for example, from Eqs. (32b) and (32c), we find

$$\langle jm'\alpha'|jm\alpha\rangle$$
$$=(a^j{}_m(\alpha)\,a^j{}_{m'}(\alpha'))^{-1}\left[\frac{(j+m)!\,(j-m')!}{(j-m)!\,(j+m')!}\right]^{1/2}$$
$$\times\sum_{nrs}(-1)^{n-m+r+s}2^{-2j+r+s}\exp[n(\theta+\theta'^*)]$$
$$\times\frac{(2j-r)!\,(2j-s)!}{r!\,(j+m-r)!\,(j-n-r)!\,s!\,(j-m'-s)!\,(j+n-s)!}. \tag{33}$$

Though the above expression is not *manifestly* symmetrical under the interchange $m\leftrightarrow m'$ (a symmetrical expression is obtained using the same representation for both $|jm\alpha\rangle$ and $|jm'\alpha'\rangle$), yet it is useful in establishing its relationship with the expression given earlier in Eq. (28b) when multiplied with $[a^j{}_m(\alpha)\,a^j{}_{m'}(\alpha')]^{-1}$. Indeed, we can perform the n-summation immediately. [Note that n in Eq. (33) need not be an integer though $j\pm n$, $m\pm n$ are so.] This allows us to rewrite the above results as

$$\langle jm'\alpha'|jm\alpha\rangle$$
$$=(a^j{}_m(\alpha)\,a^j{}_{m'}(\alpha'))^{-1}\left[\frac{(j+m)!\,(j-m')!}{(j-m)!\,(j+m')!}\right]^{1/2}$$
$$\times\sum_{rs}(-1)^{j+m-r}\exp[-(j-s)(\theta+\theta'^*)]$$
$$\times[\tfrac{1}{2}(1-\exp(\theta+\theta'^*))]^{2j-r-s}$$
$$\times\frac{(2j-r)!\,(2j-s)!}{r!\,(j+m-r)!\,s!\,(j-m'-s)!\,(2j-r-s)!}. \tag{34}$$

Using the transformation

$$F_2(\alpha,\beta,\beta',\alpha,\alpha;x,y)$$
$$=(1-x)^{-\beta}(1-y)^{-\beta'}{}_2F_1\left(\beta,\beta;\alpha;\frac{xy}{(1-x)(1-y)}\right),$$

where

$$F_2(\alpha,\beta,\beta',\gamma,\gamma';x,y)=\sum_{m,n}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!\,n!\,(\gamma)_m(\gamma')_n}x^my^n$$

and ${}_2F_1$ is the usual hypergeometrical function, [7] we reproduce the expression for $\langle jm'\alpha'|jm\alpha\rangle$ mentioned above.

V. COMPUTATION OF THE MATRIX ELEMENTS BETWEEN QUASI-INTELLIGENT STATES

In this section, we present a method, essentially based on the knowledge of a generating function, for computing matrix elements of polynomial functions of the infinitesimal generators of the rotation group between quasi-intelligent states. We first define a generating function

$$G(j,m_1,m_2;\alpha_1,\alpha_2;a,b,c)$$
$$=\langle jm_2\alpha_2|\exp\{c[J_3'(\alpha_2)]^\dagger\}\exp(bJ_3)\exp[aJ_3'(\alpha_1)]|jm_1\alpha_1\rangle. \tag{35}$$

Using

$$J_3'(\alpha)|jm\alpha\rangle=m|jm\alpha\rangle,$$

and its adjoint[8]

$$\langle jm\alpha|[J_3'(\alpha)]^\dagger=\langle jm\alpha|m,$$

we can rewrite the above generating function as

$$G(j,m_1,m_2;\alpha_1,\alpha_2;a,b,c)$$
$$=\exp(am_1+cm_2)\langle jm_2\alpha_2|\exp(bJ_3)|jm_1\alpha_1\rangle. \tag{36}$$

Now we use the method presented in Sec. III to arrive at [see Eq. (28)]

$$G(j,m_1,m_2;\alpha_1,\alpha_2;a,b,c)$$
$$=\frac{\exp(am_1+cm_2)}{a^j{}_{m_1}(\alpha_1)\,a^j{}_{m_2}(\alpha_2)}[(j+m_1)!\,(j-m_1)!\,(j+m_2)!\,(j-m_2)!]^{1/2}$$
$$\times(-1)^{2j+m_1+m_2}\sum_r[\cosh\tfrac{1}{2}(\theta_1+\theta_2^*+b)]^{-m_1-m_2+2r}$$
$$\times[\sinh\tfrac{1}{2}(\theta_1+\theta_2^*+b)]^{2j+m_1+m_2-2r}/r!\,(j+m_1-r)!$$
$$\times(j+m_2-r)!\,(-m_1-m_2+r)! \tag{37a}$$
$$=\frac{\exp(am_1+cm_2)}{a^j{}_{m_1}(\alpha_1)\,a^j{}_{m_2}(\alpha_2)}\left[\frac{(j+m_1)!\,(j-m_2)!}{(j-m_1)!\,(j+m_2)!}\right]^{1/2}$$
$$\times[\sinh\tfrac{1}{2}(\theta_1+\theta_2^*+b)]^{m_2-m_1}(-1)^{2j+m_1+m_2}$$
$$\times\sum_r(-1)^r\frac{(2j-r)!\,[\cosh\tfrac{1}{2}(\theta_1+\theta_2^*+b)]^{2j+m_1-m_2-2r}}{r!\,(j+m_1-r)!\,(j-m_2-r)!}. \tag{37b}$$

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This generating function immediately gives the matrix element

$$\langle jm\alpha_2|[(J_3'(\alpha_2))^\dagger]^{n_2}[J_3]^n[J_3'(\alpha_1)]^{n_1}|jm_1\alpha_1\rangle$$

$$= \frac{m_1^{n_1}m_2^{n_2}}{a^j{}_{m_1}(\alpha_1)\,a^j{}_{m_2}(\alpha_2)}[(j+m_1)!\,(j-m_1)!\,(j+m_2)!\,(j-m_2)!]^{1/2}$$

$$\times(-1)^{2j+m_1+m_2}\frac{\partial^n}{\partial\theta_1^n}\Bigg[\sum_r[\cosh\tfrac{1}{2}(\theta_1+\theta_2^*)]^{-m_1-m_2+2r}$$

$$\times[\sinh\tfrac{1}{2}(\theta_1+\theta_2^*)]^{2j+m_1+m_2-2r}/r!\,(j+m_1-r)!$$

$$\times(j+m_2-r)!\,(-m_1-m_2+r)!\Bigg] \tag{38}$$

In particular,

$$\langle j_2m_2\alpha_2|J_3|j_1m_1\alpha_1\rangle$$

$$=\frac{[(j+m_1)!\,(j-m_1)!\,(j+m_2)!\,(j-m_2)!]^{1/2}}{a^j{}_{m_1}(\alpha_1)\,a^j{}_{m_2}(\alpha_2)}(-1)^{2j+m_1+m_2}$$

$$\times\sum_r\{[\cosh\tfrac{1}{2}(\theta_1+\theta_2^*)]^{-m_1-m_2+2r}$$

$$\times[\sinh\tfrac{1}{2}(\theta_1+\theta_2^*)]^{2j+m_1+m_2-2r}\}$$

$$\times[r!\,(j+m_1-r)!\,(j+m_2-r)!\,(-m_1-m_2+r)!]^{-1}$$

$$\times\tfrac{1}{2}[(-m_1-m_2+2r)\tanh\tfrac{1}{2}(\theta_1+\theta_2^*)$$

$$+(2j+m_1+m_2-2r)\coth\tfrac{1}{2}(\theta_1+\theta_2^*)]. \tag{39}$$

The diagonal matrix elements of J_3 are much more interesting. We find

$$\langle jm\alpha|J_3|jm\alpha\rangle$$

$$=-j\cos\delta\Bigg[1+\frac{1}{j}$$

$$\times\frac{\sum_r\sin^2\delta(\cos\delta)^{2r-2}/r!\,(r-1)!\,(j+m-r)!\,(j-m-r)!}{\sum_r(\cos\delta)^{2r}/(r!)^2(j+m-r)!\,(j-m-r)!}\Bigg], \tag{40}$$

where we have used the notation

$$\tan\tfrac{1}{2}\delta=|\tau|=|e^\theta| \tag{41a}$$

in terms of which

$$\tanh(\tfrac{1}{2}(\theta+\theta^*))=-\cos\delta, \tag{41b}$$

$$\cosh^2(\tfrac{1}{2}(\theta+\theta^*))=(\sin\delta)^{-2}. \tag{41c}$$

Equation (40) above is a generalization of the special results given in Eq. (40) of Ref. 1 with δ in our result corresponding to the θ in this reference. One has only to go through the two calculations to appreciate the simplicity and clarity of our methods. Also we have manifested the part which goes to zero when $\delta\to0$, i.e., when $\alpha\to1$ when m can only take the value $-j$ and the state $|jm\alpha\rangle$ can be none but the Wigner state $|j(-j)\rangle$. Incidentally this also shows that the results in Ref. 1 are wrong by a factor of 2.

Next we calculate the matrix elements of J_1 and J_2 between the quasi-intelligent states.

From

$$\langle jm_2\alpha_2|J_1-i\alpha_1J_2|jm_1\alpha_1\rangle=m_1(1-\alpha_1{}^2)^{1/2}=\frac{2m_1\tau_1}{1+\tau_1^2} \tag{42a}$$

and

$$\langle jm_2\alpha_2|J_1+i\alpha_2^*J_2|jm_1\alpha_1\rangle=m_2(1-\alpha_2{}^2)^{1/2*}=\frac{2m_2\tau_2^*}{1+\tau_2^{2*}}, \tag{42b}$$

where

$$\alpha_1=\frac{1-\tau_1^2}{1+\tau_1^2}\ \text{and}\ \alpha_2=\frac{1-\tau_2^2}{1+\tau_2^2},$$

we conclude

$$\langle jm_2\alpha_2|J_1|jm_1\alpha_1\rangle$$

$$=\frac{m_1\alpha_2^*(1-\alpha_1{}^2)^{1/2}+m_2\alpha_1(1-\alpha_2{}^2)^{1/2*}}{\alpha_1+\alpha_2^*} \tag{43a}$$

$$=\frac{m_1\tau_1(1-\tau_2^{2*})+m_2\tau_2^*(1-\tau_1^2)}{1-\tau_1^2\tau_2^{2*}} \tag{43b}$$

and

$$\langle jm_2\alpha_2|J_2|jm_1\alpha_1\rangle$$

$$=i\,\frac{m_1(1-\alpha_1{}^2)^{1/2}-m_2(1-\alpha_2{}^2)^{1/2*}}{\alpha_1+\alpha_2^*} \tag{44a}$$

$$=i\,\frac{m_1\tau_1(1+\tau_2^{2*})-m_2\tau_2^*(1+\tau_1^2)}{1-\tau_1^2\tau_2^{2*}}. \tag{44b}$$

In particular

$$\langle jm\alpha_2|J_1|jm\alpha_1\rangle=\frac{m(\tau_1+\tau_2^*)}{1+\tau_1\tau_2^*} \tag{45a}$$

and

$$\langle jm\alpha_2|J_2|jm\alpha_1\rangle=i\,\frac{m(\tau_1-\tau_2^*)}{1+\tau_1\tau_2^*}. \tag{45b}$$

If we also wish to consider the special case where $\alpha_1=\alpha_2$, we shall have

$$\langle jm\alpha|J_1|jm\alpha\rangle=2m\,\frac{\mathrm{Re}\,\tau}{1+|\tau|^2} \tag{46a}$$

and

$$\langle jm\alpha|J_2|jm\alpha\rangle=-2m\,\frac{\mathrm{Im}\,\tau}{1+|\tau|^2}. \tag{46b}$$

In Eq. (18) of Ref. 1 we note the factors $\mathrm{Re}\,\tau$ and $\mathrm{Im}\,\tau$. The remaining part of the above very simple answer is hidden in uncomputed derivatives of certain generating functions. With very little work, we have been able to compute even the nondiagonal matrix elements. We remark that since J_1 and J_2 are Hermitian, our matrix elements in Eq. (44) must be invariant under the combined operation of $m_1\alpha_1\leftrightarrow m_2\alpha_2$ and complex conjugation and indeed they are evidently so. The same property demands that the matrix elements in Eq. (46) be real and these are as expected.

As another example of our method, we now obtain the matrix elements of J_1^2, $J_1J_2+J_2J_1$, and J_2^2. We shall be able to express these matrix elements in terms of those of J_3 which have been given earlier in Eq. (39).

Starting from

$$\langle jm_2\alpha_2|(J_1-i\alpha_1J_2)^2|jm_1\alpha_1\rangle=m_1^2(1-\alpha_1^2), \tag{47a}$$

$$\langle jm_2\alpha_2|(J_1+i\alpha_2{}^*J_2)^2|jm_1\alpha_1\rangle=m_2^2(1-\alpha_2{}^2)^*, \tag{47b}$$

and

$$\langle jm_2\alpha_2|(J_1+i\alpha_2{}^*J_2)(J_1-i\alpha_1J_2)|jm_1\alpha_1\rangle$$

$$=m_1m_2(1-\alpha_1{}^2)^{1/2}(1-\alpha_2{}^2)^{1/2*} \tag{47c}$$

we arrive at

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$$\langle jm_2\alpha_2|J_1^2|jm_1\alpha_1\rangle = \left[\frac{m_1(1-\alpha_1{}^2)^{1/2}\alpha_2{}^* + m_2\alpha_1(1-\alpha_2{}^2)^{1/2*}}{\alpha_1+\alpha_2{}^*}\right]^2 - \frac{\alpha_1\alpha_2{}^*}{\alpha_1+\alpha_2{}^*}\langle jm_2\alpha_2|J_3|jm_1\alpha_1\rangle, \tag{48a}$$

$$\langle jm_2\alpha_2|\tfrac{1}{2}(J_1J_2+J_2J_1)|jm_1\alpha_1\rangle = i[(m_1(1-\alpha_1{}^2)^{1/2} - m_2(1-\alpha_2{}^2)^{1/2*})(m_1\alpha_2{}^*(1-\alpha_1{}^2)^{1/2} + m_2\alpha_1(1-\alpha_2{}^2)^{1/2*})]/\alpha_1+\alpha_2{}^* - \frac{i}{2}\,\frac{\alpha_1-\alpha_2{}^*}{\alpha_1+\alpha_2{}^*}\langle jm_2\alpha_2|J_3|jm_1\alpha_1\rangle \tag{48b}$$

and

$$\langle jm_2\alpha_2|J_2{}^2|jm_1\alpha_1\rangle = -\left[\frac{m_1(1-\alpha_1{}^2)^{1/2} - m_2(1-\alpha_2{}^2)^{1/2*}}{\alpha_1+\alpha_2{}^*}\right]^2 - \frac{1}{\alpha_1+\alpha_2{}^*}\langle jm_2\alpha_2|J_3|jm_1\alpha_1\rangle. \tag{48c}$$

Note that the parts on the right-hand side of Eqs. (48) above, which are independent of J_3, could have been obtained from the matrix elements of J_1 and J_2 given earlier in Eqs. (43) and (44), as should be the case, since these are the values of the corresponding matrix elements *provided* J_1 and J_2 commuted. Also defining

$$\langle jm_2\alpha_2|\Delta J_i^2|jm_1\alpha_1\rangle = \langle jm_2\alpha_2|J_i^2|jm_1\alpha_1\rangle - (\langle jm_2\alpha_2|J_i|jm_1\alpha_1\rangle)^2 \tag{49}$$

for $i=1,2$, we obtain

$$\langle jm_2\alpha_2|\Delta J_1^2|jm_1\alpha_1\rangle = -\frac{\alpha_1\alpha_2{}^*}{\alpha_1+\alpha_2{}^*}\langle jm_2\alpha_2|J_3|jm_1\alpha_1\rangle \tag{50a}$$

and

$$\langle jm_2\alpha_2|\Delta J_2^2|jm_1\alpha_1\rangle = -\frac{1}{\alpha_1+\alpha_2{}^*}\langle jm_2\alpha_2|J_3|jm_1\alpha_1\rangle. \tag{50b}$$

In the above results, the m-dependence of the matrix elements on the left is *entirely* given in terms of the m-dependence of the matrix elements of J_3.

From Eqs. (50), we conclude

$$\langle jm_2\alpha_2)|\Delta J_1^2|jm_1\alpha_1\rangle\langle jm_2\alpha_2|\Delta J_2^2|jm_1\alpha_1\rangle = \frac{\alpha_1\alpha_2{}^*}{(\alpha_1+\alpha_2{}^*)^2}(\langle jm_2\alpha_2|J_3|jm_1\alpha_1\rangle)^2. \tag{51}$$

Considering the diagonal matrix elements, the above result implies

$$\langle jm\alpha|\Delta J_1^2|jm\alpha\rangle\langle jm\alpha|\Delta J_2^2|jm\alpha\rangle = \frac{|\alpha|^2}{4|\mathrm{Re}\,\alpha|^2}(\langle jm\alpha|J_3|jm\alpha\rangle)^2, \tag{52}$$

which will be $\frac{1}{4}(\langle jm\alpha|J_3|jm\alpha\rangle)^2$ *only when* α *is real.* Thus we have verified that from amongst the quasi-intelligent states, only the intelligent states satisfy equality in the Heisenberg uncertainty relation and for all other quasi-intelligent states

$$\Delta J_1^2\cdot\Delta J_2^2 > \tfrac{1}{4}(\langle J_3\rangle)^2$$

as expected.

Since in the above example, our interest was first to exemplify the efficiency of our method and second to reproduce equality in the Heisenberg uncertainty relation as a check on our methods, we have not tried to express our results in terms of the τ's though it could be done trivially.

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[1]C. Aragone, E. Chalbaud, and S. Salamo, J. Math. Phys. **17**, 1963 (1976).

[2]See, for example, E. Merzbacher, in *Quantum Mechanics* (Wiley, New York, 1970), pp. 159—60.

[3]The normalization of the operators $J_3'(\alpha)$ and $J_\pm'(\alpha)$ have been chosen so as to satisfy Eqs. (15) and (16) which are exactly the same as satisfied by J_3, $J_\pm$. As one consequence of this normalization, we find that the spectrum of $J_3'(\alpha)$ for $\alpha\neq\pm 1$ coincides with the spectrum of J_3 and thus, for a given j, consists of the m's which satisfy $-j\leq m\leq j$ with unit steps.

[4]$\alpha=0$ is a point which is simultaneously purely real and purely imaginary. Again $\alpha=0\Longrightarrow\tau\Longrightarrow 1\Longrightarrow\theta=2n\pi i$ and hence the corresponding states $|jm0\rangle$ are indeed obtainable from $|jm\rangle$ by a physical rotation. But this result is obvious since with $\alpha=0$, $J_3'(0)=J_1$ which could obviously be obtained from J_3 by rotation. This special case has no features different from those of the Wigner states and is ignored in the sequel.

[5]See, e.g., footnote 8 in Ref. 1.

[6]The usual representation for the σ matrices is given by

$$\sigma_1=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad \sigma_2=\begin{bmatrix}0&-i\\i&0\end{bmatrix},\quad \sigma_3=\begin{bmatrix}1&0\\0&-1\end{bmatrix}.$$

[7]H. Bateman in *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. I, p. 238, Eq. (3).

[8]$J_3'(\alpha)=\dfrac{J_1-i\alpha J_2}{(1-\alpha^2)^{1/2}}\Longrightarrow[J_3'(\alpha)]^\dagger=\dfrac{J_1+i\alpha^*J_2}{[(1-\alpha^2)^{1/2}]^*}.$

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