Minimum Uncertainty Measurements of Angle and Angular Momentum

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(Received 10 May 2006; published 15 December 2006)

We present an accurate description of the conjugate pair angle–angular momentum in terms of the exponential of the angle instead of the angle itself, which leads to dispersion as a natural measure of resolution. Intelligent states minimizing the uncertainty product under the constraint of a given uncertainty in angle or in angular momentum turn out to be given by Mathieu wave functions. We discuss Gaussian approximations to these optimal states in terms of von Mises distributions. The theory is successfully applied to the spatial degrees of freedom of a photon and verified in an experiment that employs computer-controlled spatial light modulators at both the state preparation and the analyzing stages.

DOI: 10.1103/PhysRevLett.97.243601 PACS numbers: 42.50.Vk, 03.67.−a, 42.50.Dv

Light carries and transfers energy as well as linear and angular momentum. The angular momentum contains a spin contribution, associated with polarization, and an orbital component, linked with the spatial profile of the light intensity and phase [1]. The seminal paper of Allen *et al.* [2] firmly established that the Laguerre-Gauss modes, typical of cylindrical symmetry, carry a well-defined angular momentum per photon. In the paraxial limit, this orbital component is polarization independent and arises solely from the azimuthal phase dependence $e^{im\phi}$, which gives rise to spiral wave fronts. The index m takes only integer values and can be seen as the eigenvalue of the orbital angular-momentum operator. In consequence, the Laguerre-Gauss modes constitute a complete set and can be used to represent quantum photon states [3–5].

The possibility of using these light fields for driving micromachines, such as optical tweezers or traps, has attracted a good deal of attention [6–8]. Besides, entangled photons prepared in a superposition of states bearing a well-defined orbital angular momentum provide access to multidimensional entanglement. This is of considerable importance in quantum information and cryptography, because with these states more information can be stored and there is less sensitivity to decoherence [9–12].

In the following, we deal with cylindrical symmetry: We are concerned with the planar rotations by an angle ϕ in the plane x-y, generated by the angular momentum along the z axis, which for simplicity will be denoted henceforth as $\hat{L}$. In this respect, we wish to bring up that the proper definition of angular variables in quantum mechanics is beset by well-known difficulties [13,14]. For the case of a harmonic oscillator, the problems essentially arise from two basic sources: the periodicity and the semiboundedness of the energy spectrum. The first prevents the existence of a phase operator but not of its exponential. The second entails that this exponential is not unitary.

Although we have here the same kind of problems linked with the periodicity, the angular momentum has a spectrum that includes both positive and negative integers, which allows us to introduce a well-behaved exponential of the angle operator, denoted by $\hat{E}$. Since the angle is canonically conjugate to $\hat{L}$, we start from the commutation relation [15,16]

$$[\hat{E}, \hat{L}] = \hat{E}. \tag{1}$$

The goal of this Letter is to develop the first comprehensive approach to the minimum uncertainty states for the relation (1). Previous attempts to deal with these matters run into difficulties stemming from the fact that a periodic variable—angle—is described by a nonperiodic measure of spread—variance—[17,18]. Consequently, the angular uncertainty depends on the 2π window chosen. Furthermore, the conjugate variable to the angular momentum is treated heuristically. For example, the angular uncertainty is delimited by a wedge structure [19], and there is no quantum counterpart of the angle in the measurement scheme. Finally, the associated commutation relation depends on the value of the angle distribution at a point, which makes the uncertainty relation and the corresponding minimum uncertainty states cumbersome.

Our results strengthen the evidence that $\hat{E}$ furnishes a correct description of the angle. The associated (constrained) intelligent states prove to be the Mathieu functions when a meaningful periodic resolution measure (namely, the dispersion) is employed. We shall also bring out that $\hat{E}$ can be associated with a feasible transformation (a forklike hologram) that shifts the values of the angular momentum. Though this transformation is widely used for holographic detection of vortex beams, its fundamental role as the conjugate variable to the angular momentum has not yet been recognized. Therefore, the minimum uncertainty states are delimiting borders for these quantum variables, and our formulation thus paves the way for a full quantum processing of vortex beams. In this sense, this Letter provides not only the first rigorous fully quantum formulation of the angular momentum and its conjugate variable but also a bridge between the well-developed classical theory of singular optics and the realm of quantum optics.

Let us start by recalling that in directional statistics a simple meaningful measure of angular spread is the dispersion (or circular variance) [20,21], defined as $\sigma_\phi^2 = 1 - |\langle e^{i\phi}\rangle|^2$, where the average values are computed with the angle distribution $P(\phi)$. In our case, this coincides with $(\Delta\hat{E})^2 = \langle \hat{E}^\dagger \hat{E}\rangle - \langle \hat{E}^\dagger\rangle\langle\hat{E}\rangle$, which is an adapted version of the variance for unitary operators [22]. As expected, it possesses all of the good properties: It is periodic, the shifted distributions $P(\phi + \phi')$ are characterized by the same resolution, and for sharp angle distributions it coincides with the standard variance since $|\langle e^{i\phi}\rangle|^2 \simeq 1 + \langle\phi^2\rangle$. This confirms that the statistics of $\hat{E}$ should be considered instead of the angle itself.

The action of the unitary operator $\hat{E}$ in the angular-momentum basis is

$$\hat{E}|m\rangle = |m - 1\rangle, \tag{2}$$

where the integer m runs from $-\infty$ to $+\infty$. Therefore, $\hat{E}$ possesses a simple optical implementation by means of a phase mask removing a charge $+1$ from a vortex beam. In the representation generated by the normalized eigenvectors of $\hat{E}$, we can write $\hat{L} \mapsto -id/d\phi$ and $\hat{E} \mapsto e^{i\phi}$, which formally verify the fundamental relation (1).

Let us turn to the corresponding uncertainty relation. When the standard form $(\Delta\hat{A})^2(\Delta\hat{B})^2 \geq \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2$ is applied to Eq. (1) and the previous notion of dispersion is used, we get

$$(\Delta\hat{E})^2(\Delta\hat{L})^2 \geq \tfrac{1}{4}[1 - (\Delta\hat{E})^2]. \tag{3}$$

This can be recast in terms of the cosine and sine operators $\hat{C} = (\hat{E} + \hat{E}^\dagger)/2$ and $\hat{S} = i(\hat{E}^\dagger - \hat{E})/2$, yielding

$$(\Delta\hat{C})^2(\Delta\hat{L})^2 \geq \tfrac{1}{4}|\langle\hat{S}\rangle|^2, \qquad (\Delta\hat{S})^2(\Delta\hat{L})^2 \geq \tfrac{1}{4}|\langle\hat{C}\rangle|^2. \tag{4}$$

States satisfying the equality in an uncertainty relation are sometimes referred to as intelligent states [23]. However, in the case of Eq. (3), the inequality cannot be saturated [24], since this would imply to saturate both relations in (4) simultaneously. In other words, the formulation (3) is true but too weak.

To get a saturable lower bound, we look instead at normalized states that minimize the uncertainty product $(\Delta\hat{E})^2(\Delta\hat{L})^2$ either for a given $(\Delta\hat{E})^2$ or for a given $(\Delta\hat{L})^2$. We approach this problem by the method of undetermined multipliers. The linear combination of variations lead to the basic equation

$$[\hat{L}^2 + \mu\hat{L} + (q^*\hat{E} + qE^\dagger)/2]|\Psi\rangle = a|\Psi\rangle, \tag{5}$$

where μ, q, and a are Lagrange multipliers. We shall solve this eigenvalue equation in the angle representation $\Psi(\phi) = \langle\phi|\Psi\rangle$. Note first that, without loss of generality, we can restrict ourselves to states with $\langle\hat{L}\rangle = 0$, since we readily obtain solutions with mean angular momentum $\bar{m}$ by multiplying the wave function by $\exp(i\bar{m}\phi)$. Alternatively, we observe that the change of variables $\exp(i\mu\phi)\Psi(\phi)$ eliminates the linear term from (5). In addition, we can take q to be a real number, since this introduces only an unessential global phase shift. To properly interpret this eigenvalue problem, we also introduce the rescaled angular variable $\eta = \phi/2$. Surprisingly, this turns Eq. (5) into the standard form of the Mathieu equation

$$\frac{d^2\Psi(\eta)}{d\eta^2} + [a - 2q\cos(2\eta)]\Psi(\eta) = 0. \tag{6}$$

Let us note in passing that Mathieu states have many applications not only in optics but also in other branches of modern physics [25]. The variable η has a domain $0 \leq \eta < 2\pi$ and plays the role of polar angle in elliptic coordinates. In our case, the required periodicity imposes that the only acceptable Mathieu functions are those periodic with a period of π or 2π. The values of a in Eq. (6) that satisfy this condition are the eigenvalues of this equation. We have then two families of independent solutions, namely, the angular Mathieu functions $\mathrm{ce}_n(\eta, q)$ and $\mathrm{se}_n(\eta, q)$, with $n = 0, 1, 2, \ldots$, which are usually known as the elliptic cosine and sine, respectively. The parity of these functions is exactly the same as their trigonometric counterparts; that is, $\mathrm{ce}_n(\eta, q)$ is even and $\mathrm{se}_n(\eta, q)$ is odd in η, while they have period π when n is even or period 2π when n is odd.

Since the 2π periodicity in ϕ requires π periodicity in η, the acceptable solutions for our eigenvalue problem are the independent Mathieu functions of even order

$$\Psi_{2n}(\eta, q) = \sqrt{\frac{2}{\pi}} \times \begin{cases} \mathrm{ce}_{2n}(\eta, q), \\ \mathrm{se}_{2n}(\eta, q), \end{cases} \qquad n = 0, 1, \ldots, \tag{7}$$

where the numerical factor ensures a proper normalization. In what follows, we consider only even solutions $\mathrm{ce}_{2n}(\eta, q)$, although a parallel treatment can be done for the odd ones. After some calculations, we get

$$\begin{aligned} (\Delta\hat{L})^2_{2n} &= \tfrac{1}{4}[A_{2n}(q) - 2q\Theta_{2n}(q)], \\ (\Delta\hat{E})^2_{2n} &= 1 - |\Theta_{2n}(q)|^2, \end{aligned} \tag{8}$$

with $\Theta_{2n}(q) = A_0^{(2n)}(q)A_2^{(2n)}(q) + \sum_k A_{2k}^{(2n)}(q)A_{2k+2}^{(2n)}(q)$, and the coefficients $A_{2k}^{(2n)}$ are the Fourier components of the periodic function $\mathrm{ce}_{2n}(\eta, q)$.

If we expand $\mathrm{ce}_{2n}(\eta, q)$ in powers of q and retain only linear terms [25], we have

$$(\Delta\hat{L})^2_{2n} = \frac{(2n)^2}{4} + \frac{4n^4 - 3n^2 + 1}{8(4n^2-1)^2} q^2,$$
$$(\Delta\hat{E})^2_{2n} = 1 - \frac{1}{4(4n^2-1)^2} q^2, \tag{9}$$

which shows a quadratic increasing with q of the angular-momentum variance and a decreasing of the angle dispersion. The uncertainty product, up to terms q^2, reads as

$$(\Delta\hat{E})^2_{2n}(\Delta\hat{L})^2_{2n} = n^2 + \tfrac{1}{4}\{(4n^4 - 5n^2 + 1)[1 - (\Delta\hat{E})^2_{2n}]\}. \tag{10}$$

It is clear that this product attains its minimum value for $n = 0$, which saturates the bound in Eq. (3) for this range of values of q. Moreover, one can easily verify that this fundamental mode $n = 0$ is the minimum uncertainty state for all values of q.

We observe that for large dispersions ($q \to 0$) the fundamental wave function may be approximated by $P_0(\phi) \propto |\mathrm{ce}_0(\eta, q)|^2 \simeq \exp(-q\cos\phi)$, which is the von Mises distribution, also known as the normal distribution on the unit circle [26]. This remarkable result shows that our optimal states are very close to Gaussians on the unit circle. Curiously enough, it has been recently found that the von Mises distribution maximizes the entropy for a fixed value of the dispersion [27]. In the opposite limit of small dispersions ($q \to \infty$), one can also check that $P_0(\phi) \propto |\mathrm{ce}_0(\eta, q)|^2 \simeq \exp(-\sqrt{q}\cos\phi)$. Therefore, von Mises wave functions constitute an excellent approximation to the Mathieu wave functions, except perhaps for intermediate values of the dispersions, where a deviation may occur. In Fig. 1, we have plotted $(\Delta\hat{E})(\Delta\hat{L})$ in terms of $(\Delta\hat{E})$. The solid line represents the fundamental Mathieu beam, which provides the optimal angular resolution, while the dashed line represents the von Mises approximation. The very small difference between these two curves is magnified in the inset. For the purposes of comparison, the ideal bound coming from Eq. (3) is plotted as a dotted line. These results are free from artifacts arising from an inappropriate quantification of the angle spread. In particular, the minimum uncertainty states with large dispersions present wide angular distributions and vice versa.

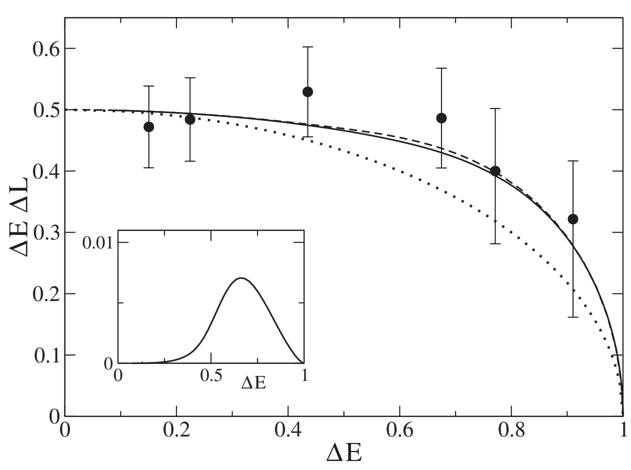

FIG. 1. Theoretical and experimental uncertainty products as a function of the dispersion. The solid line represents the fundamental Mathieu beam, while the dashed line represents the von Mises approximation. The difference between these two wave functions appears plotted in the inset. We have included also the ideal bound given by Eq. (3).

To verify our theory, we performed an experiment with the spatial degrees of freedom of light prepared in various von Mises states. Given the very small difference between the uncertainty products of the optimal Mathieu beams and their Gaussian approximations, the measurement would also reveal, as a side product, whether such a small deviation from the Gaussian character is observable with the present commercially available technology.

Figure 2 shows our setup. Two spatial light modulators (SLM) were used: The amplitude SLM (CRL Opto, 1024×768 pixels) prepares desired input states, while the phase SLM (Boulder, 512×512 pixels) works as an analyzing hologram.

The beam generated by an Ar laser (514 nm, 200 mW) is spatially filtered, expanded, and collimated by the lens L_1 and impinges on the hologram generated by the amplitude SLM. The bitmap of the hologram is computed as an interference pattern of the desired state $U_s = \sum_{m=-\infty}^{\infty} a_m e^{im\phi}$ and an inclined reference plane wave. After illuminating the hologram with the collimated beam, the Fourier spectrum of the transmitted beam is localized at the back focal plane of the first Fourier lens FL_1 and consists of three diffraction orders $(-1, 0, +1)$. The undesired 0 and -1 orders are removed by a spatial filter. After inverse Fourier transformation, performed by the second Fourier lens FL_2, a collimated beam with the required complex amplitude profile U_s is obtained. This completes the state preparation.

The analysis begins by reflecting the prepared field U_s at a phase SLM, whose reflectivity is proportional to $t \propto e^{iM\phi}$, where ϕ is the azimuthal angle. A Fourier transformation of the reflected field yields the spatial spectrum $\bar{U} \propto \sum_m a_m \exp[i(m+M)\phi] A_{m+M}(\nu)$, where $A_j(0) = 0$ for $j \neq 0$ and $A_0(0) \neq 0$. It is obvious that the component of helicity $m = -M$ gives rise to a light spot of intensity $I_0 = A_0^2 a_m^2$, while the other components do not contribute

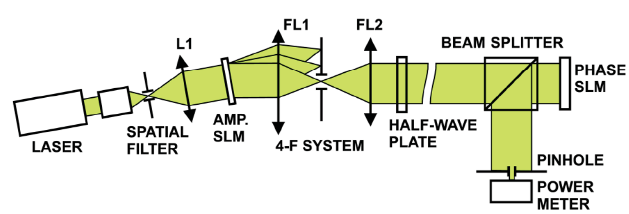

FIG. 2 (color online). Experimental setup for the generation of beams with a von Mises distribution and subsequent detection of the associated angular-momentum components.

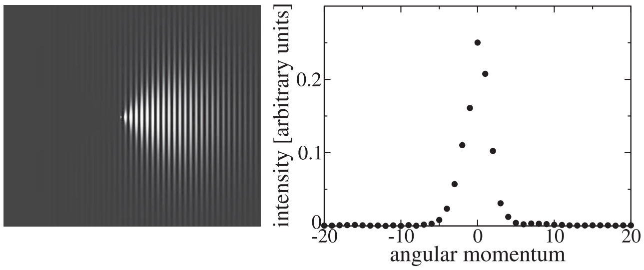

FIG. 3. Preparation and measurement of a von Mises beam of dispersion $\Delta\hat{E} = 0.435$. Left: Computed hologram; right: measured intensities of components of different helicities m.

to the measured intensity. In this way, the weight coefficients of the superposition can be determined by selective intensity measurements using a pinhole and a power meter. Of course, in the real experiment, the signal beams are transversally bounded, so the intensity of the axial point does not vanish completely when $M \neq -m$. To avoid this cross talk, calibrating response functions were acquired for each phase mask.

After the setup was carefully aligned using Laguerre-Gauss beams, von Mises distributions of transversal amplitude (differing by their angular dispersion) were generated. Each von Mises state was then scanned for values of the helicities in the range of $m \in [-20, 20]$. A typical preparation hologram and the corresponding raw measured data are shown in Fig. 3. The angular-momentum distribution was obtained by correcting these measured intensities for a nonzero width of the analyzer response function, which was measured separately for each mask. Finally, errors were estimated by fitting the measured angular-momentum distributions to the theoretically calculated distributions. The resulting experimental uncertainty products are depicted in Fig. 1 by solid circles. Given the accuracy of the measurements (indicated by error bars in Fig. 1), they fit quite well the theoretical predictions. Our present experiment distinguishes between the uncertainty product of optimal states and the ideal limit. It is, however, not possible to discriminate between the Mathieu and von Mises beams. Keeping in mind that von Mises states play the same role for the spatial degrees of freedom as Gaussian states for quadratures, the observation of the nonclassical behavior of angle and angular momentum is a challenging problem left for future studies.

In conclusion, we have formulated rigorous uncertainty relations for angle and angular momentum based on dispersion as a correct statistical measure of error. An optical test of the derived uncertainty relations was performed by using spatial light modulators for both the preparation and the analysis.

We acknowledge discussions with Hubert de Guise. This work was supported by the Czech Ministry of Education, Projects No. MSM6198959213 and No. LC06007, the Czech Grant Agency, Grant No. 202/06/307, and the Spanish Research Directorate, Grant No. FIS2005-06714.

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