Intelligent states

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Intelligent states Part 1: Introduction and simple cases

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Basic definition

Given two incompatible observables $\hat{A}$ and $\hat{B}$, $[\hat{A}, \hat{B}] \neq 0$, there exists an restriction on the possibility of preparing a state in which these two observables simultaneously take values. This restriction takes the form of an inequality called an uncertainty relation:

$$\Delta A \Delta B \geq \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle.$$

It is natural to ask if there are states for which the lower bound is reached, i.e. for which $\Delta A \Delta B = \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle$.

Definition:
An intelligent state of $\hat{A}$ and $\hat{B}$ is a normalized state $|\psi\rangle$ for which

$$\Delta A \Delta B = \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle| \tag{1}$$

$\Delta\hat{A}$

- Take $|\psi\rangle$ any normalized ket.
- Write $\langle\psi|\hat{A}|\psi\rangle = \langle A\rangle$.
- For any observable $\hat{A}$, define

$$\Delta\hat{A} \equiv \hat{A} - \langle A\rangle\,, \tag{2}$$

- Then

$$\langle(\Delta\hat{A})^2\rangle \equiv \langle\psi|\left(\hat{A} - \langle A\rangle\right)^2|\psi\rangle \tag{3}$$

- Expand:

$$\begin{aligned}\langle(\Delta\hat{A})^2\rangle &= \langle\psi|\hat{A}^2 - 2\hat{A}\langle A\rangle + \langle A\rangle^2|\psi\rangle \\ &= \langle\psi|\hat{A}^2|\psi\rangle - 2\langle\psi|\hat{A}|\psi\rangle\langle A\rangle + \langle A\rangle^2\end{aligned} \tag{4}$$

- Collect:

$$\begin{aligned}\langle(\Delta\hat{A})^2\rangle &= \langle\psi|\hat{A}^2|\psi\rangle - \langle A\rangle^2\,, \\ &= \langle A^2\rangle - \langle A\rangle^2 \\ &\equiv (\Delta A)^2 \geq 0\,.\end{aligned} \tag{5}$$

$\Delta A\,\Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

We now show the **uncertainty relation**:

$$(\Delta A)^2\ (\Delta B)^2 \geq \frac{1}{4}\,|\langle[\hat{A},\hat{B}]\rangle|^2\,. \tag{6}$$

Proof:

- Let

$$\begin{aligned} |\psi_A\rangle \equiv \Delta\hat{A}|\psi\rangle &= \hat{A}|\psi\rangle - |\psi\rangle\langle\psi|\hat{A}|\psi\rangle\,, \\ |\psi_B\rangle \equiv \Delta\hat{B}|\psi\rangle &= \hat{B}|\psi\rangle - |\psi\rangle\langle\psi|\hat{B}|\psi\rangle \end{aligned} \tag{7}$$

  where $|\psi\rangle$ is any normalized state (although $|\psi_A\rangle$ may not be normalized).
- By Schwarz inequality (a.k.a the triangle inequality),

$$\langle\psi_A|\psi_A\rangle\langle\psi_B|\psi_B\rangle \geq |\langle\psi_A|\psi_B\rangle|^2 \tag{8}$$

- We note, for future reference, that

$$\langle\psi_A|\psi_A\rangle\langle\psi_B|\psi_B\rangle = |\langle\psi_A|\psi_B\rangle|^2 \Rightarrow |\psi_A\rangle = \mu\,|\psi_B\rangle\,, \tag{9}$$

  i.e. strict equality implies $|\psi_A\rangle$ is a scalar multiple of $|\psi_B\rangle$.

$\Delta A\,\Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

From

$$\langle\psi_A|\psi_A\rangle\langle\psi_B|\psi_B\rangle \geq |\langle\psi_A|\psi_B\rangle|^2 \tag{10}$$

assume that $\hat{A}$ and $\hat{B}$ are hermitian so

$$(\Delta\hat{A})^\dagger = \Delta\hat{A}\,, \quad (\Delta\hat{B})^\dagger = \Delta\hat{B} \tag{11}$$

- Then

$$\langle\psi_A|\psi_A\rangle = \langle\psi|\left(\Delta\hat{A}\right)^\dagger\Delta\hat{A}|\psi\rangle = \langle(\Delta A)^2\rangle\,, \tag{12}$$

  and similarly for $\langle\psi_B|\psi_B\rangle$.

- Expand

$$\langle\psi_A|\psi_B\rangle = \langle\psi|\Delta\hat{A}\Delta\hat{B}|\psi\rangle \tag{13}$$

$$= \tfrac{1}{2}\langle\psi|(\Delta\hat{A}\Delta\hat{B} - \Delta\hat{B}\Delta\hat{A})|\psi\rangle + \tfrac{1}{2}\langle\psi|(\Delta\hat{A}\Delta\hat{B} + \Delta\hat{B}\Delta\hat{A})|\psi\rangle\,. \tag{14}$$

$\Delta A\, \Delta B \geq \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle|$

- Look at $\frac{1}{2}\langle\psi|(\Delta\hat{A}\Delta\hat{B} - \Delta\hat{B}\Delta\hat{A})|\psi\rangle$ and write

$$\begin{aligned} \tfrac{1}{2}\langle\psi|(\Delta A\Delta B - \Delta B\Delta A)|\psi\rangle &= \tfrac{1}{2}\langle\psi|[\Delta A, \Delta B]|\psi\rangle \\ &= \tfrac{1}{2}\langle\psi|[\hat{A}, \hat{B}]|\psi\rangle\,. \end{aligned} \tag{15}$$

- With $\hat{A}$, $\hat{B}$ hermitian, $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is anti-hermitian:

$$\begin{aligned} [\hat{A}, \hat{B}]^\dagger &= (\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger = (\hat{A}\hat{B})^\dagger - (\hat{B}\hat{A})^\dagger\,, \\ &= \hat{B}^\dagger\hat{A}^\dagger - \hat{A}^\dagger\hat{B}^\dagger \\ &= \hat{B}\hat{A} - \hat{A}\hat{B} = -[\hat{A}, \hat{B}]\,, \end{aligned} \tag{16}$$

- Hence,

$$\langle\psi|[\Delta\hat{A}, \Delta\hat{B}]|\psi\rangle = \langle\psi|[\hat{A}, \hat{B}]|\psi\rangle = ic_-\,, \qquad c_- \in \mathbb{R} \tag{17}$$

- $\Delta A\Delta B + \Delta B\Delta A$ is hermitian. Thus we write

$$\langle\psi|\Delta A\Delta B + \Delta B\Delta A|\psi\rangle = ic_+\,, \qquad c_+ \in \mathbb{R}\,. \tag{18}$$

$\Delta A \, \Delta B \geq \ \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle|$

Hence

$$\begin{aligned} \langle \psi_A | \psi_B \rangle &= \tfrac{1}{2}(ic_- + c_+)\,, \\ |\langle \psi_A | \psi_B \rangle|^2 &= \tfrac{1}{4}(c_-^2 + c_+^2) \geq \tfrac{1}{4}\ c_-^2 \\ &\geq \tfrac{1}{4}\,|\langle [\hat{A}, \hat{B}] \rangle|^2 \,. \end{aligned} \tag{19}$$

Putting it all together, we get

$$(\Delta A)^2 \ (\Delta B)^2 \geq |\langle \psi_A | \psi_B \rangle|^2 \geq \tfrac{1}{4}\,|\langle [\hat{A}, \hat{B}] \rangle|^2 \,, \tag{20}$$

which proves the claim.

$\Delta A\,\Delta B = \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

To replace

$$\Delta A\,\Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle| \;\rightarrow\; \Delta A\,\Delta B = \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle| \tag{21}$$

we must satisfy two conditions:

- $|\psi_A\rangle = \mu\,|\psi_B\rangle$ i.e.

$$\Delta\hat{A}|\psi\rangle = \mu\,\Delta\hat{B}|\psi\rangle \tag{22}$$

- $c_+ = \langle\psi|\left(\Delta\hat{A}\Delta\hat{B} + \Delta\hat{B}\Delta\hat{A}\right)|\psi\rangle = 0.$
  - Using Eq.(22) and its conjugate:

$$c_+ = 0 = \mu^*\langle\psi|(\Delta\hat{B})^2|\psi\rangle + \mu\langle\psi|(\Delta\hat{B})^2|\psi\rangle \tag{23}$$

  - But $\langle\psi|(\Delta\hat{B})^2|\psi\rangle$ is real so $\mu^* = -\mu$, or $\mu = i\alpha$, with $\alpha \in \mathbb{R}$.
- Hence we get the condition on $|\psi\rangle$:

$$\left(\hat{A} - i\alpha\hat{B}\right)|\psi\rangle = \lambda|\psi\rangle\,, \qquad \lambda = \langle A\rangle - i\alpha\langle B\rangle\,. \tag{24}$$

$\Delta A\, \Delta B = \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle|$

Moreover:

$$\langle\psi| \left(\Delta\hat{A}\right)^2 |\psi\rangle = |\mu|^2 \langle\psi| \left(\Delta\hat{B}\right)^2 |\psi\rangle = \alpha^2 \langle\psi| \left(\Delta\hat{B}\right)^2 |\psi\rangle \tag{25}$$

Thus,

$$(\Delta A)^2 = \alpha^2 (\Delta B)^2 \tag{26}$$

$\hat{x}$ and $\hat{p}$

- Rearrange

$$0 = \langle\psi|\Delta\hat{x}\Delta\hat{p}|\psi\rangle + \langle\psi|\Delta\hat{p}\Delta\hat{x}|\psi\rangle \tag{27}$$

using $\Delta\hat{x}\Delta\hat{p} = \Delta\hat{p}\Delta\hat{x} + i\,\mathbb{1}$ to get:

$$2\langle\psi|\Delta\hat{p}\Delta\hat{x}|\psi\rangle = -i \tag{28}$$

$$2\langle\psi|\Delta\hat{x}\Delta\hat{p}|\psi\rangle = +i\,. \tag{29}$$

- Use $\Delta\hat{x}|\psi\rangle = i\alpha\Delta\hat{p}|\psi\rangle$ to get

$$2i\alpha\langle\psi|\left(\Delta\hat{p}\right)^2|\psi\rangle = -i \Rightarrow \left(\Delta p\right)^2 = -\frac{1}{2\alpha} \tag{30}$$

and

$$-\frac{2i}{\alpha}\langle\psi|\left(\Delta\hat{x}\right)^2|\psi\rangle = i \Rightarrow \left(\Delta x\right)^2 = -\frac{\alpha}{2}\,. \tag{31}$$

- Thus α is further constrained to $\alpha < 0$, with

$$\alpha = -\Delta x/\Delta p\,. \tag{32}$$

$\hat{x}$ and $\hat{p}$

- Recall from basic quantum mechanics ($\hbar = 1$):

$$\langle x|\hat{x}|\psi\rangle \mapsto x\psi(x)\,, \quad \langle x|\hat{p}|\psi\rangle \mapsto i\frac{d}{dx}\psi(x)\,. \tag{33}$$

- Thus, we get

$$\begin{aligned} \langle x|\left(\hat{x} - i\alpha\hat{p}\right)|\psi\rangle \quad \mapsto \quad & \left(x + \alpha\frac{d}{dx}\right)\psi(x) \\ & = \left(\langle x\rangle - i\alpha\langle p\rangle\right)\,\psi(x) \end{aligned} \tag{34}$$

- Rearrange:

$$\frac{d\psi(x)}{\psi(x)} = \frac{1}{\alpha}(\langle x\rangle - i\alpha\langle p\rangle - x)dx \tag{35}$$

- which yields:

$$\psi(x) \quad = \quad C\,\mathrm{e}^{\frac{1}{\alpha}(\langle x\rangle x - \frac{1}{2}x^2) - i\langle p\rangle\, x}\,, \tag{36}$$

$$= \quad \tilde{C}\,\mathrm{e}^{\frac{1}{2\alpha}(x-\langle x\rangle)^2 - i\langle p\rangle x} \tag{37}$$

$\hat{x}$ and $\hat{p}$

Notes:

- $\langle x \rangle$, $\langle p \rangle$ and α are parameters; thus we rightly write

$$\psi(x) \to \psi(x; \langle x \rangle, \langle p \rangle, \alpha) \tag{38}$$

- Since $\alpha < 0$, $\psi(x; \langle x \rangle, \langle p \rangle, \alpha)$ is normalizable over the real line.
- For $\alpha = -1$, we have $\Delta x = \Delta p$ (in suitable units) and $\psi(x; \langle x \rangle, \langle p \rangle, -1)$ is just a displaced harmonic oscillator ground state.

Self-adjoint *vs* hermitian

There is a technical difference between self–adjoint and hermitian operators. The difference becomes important when looking at the (infinite–dimensional) space of functions. We illustrate as follows.

- Suppose an operator $\hat{A}$ acts on a subset of "legal" kets, which are contained in some space $\mathfrak{H}$.
- Let $|\phi\rangle, |\zeta\rangle$ two vectors in $\mathfrak{H}$ so that

$$\langle \phi \,|\, \hat{A}\psi \rangle = \langle \zeta \,|\, \psi \rangle \tag{39}$$

  for any "legal" ket $|\psi\rangle$.
- Then we write

$$\hat{A}^\dagger |\phi\rangle = |\zeta\rangle \tag{40}$$

  and $\hat{A}^\dagger$ is the adjoint of $\hat{A}$.
- $\hat{A}^\dagger$ is well defined on all the vectors $|\phi\rangle$ for which $\hat{A}^\dagger|\phi\rangle = |\zeta\rangle$ is valid.
- An operator is self–adjoint if $\hat{A}^\dagger = \hat{A}$. In particular, this implies the set of "legal" kets for $\hat{A}^\dagger$ coincides with the set of "legal" kets for $\hat{A}$.

$\hat{\varphi}$ is not self–adjoint

- Consider the pair:

$$\hat{\varphi} \mapsto \varphi\,, \qquad \hat{L}_z \mapsto i\frac{d}{d\varphi} \tag{41}$$

- These act on periodic functions $f(\varphi) = f(\varphi + 2\pi)$ as

$$\hat{\varphi}\, f(\varphi) = \varphi\, f(\varphi)\,, \qquad \hat{L}_z\, f(\varphi) = i\frac{d}{d\varphi} f(\varphi)\,. \tag{42}$$

- $\varphi\, f(\varphi)$ is no longer periodic, *i.e.* is no longer a "legal" function.
- Thus, $\hat{\varphi}$ cannot be self–adjoint.

$\hat{L}_z$ and $\mathrm{e}^{i\varphi}$

- Introduce the operator $\mathrm{e}^{i\varphi}$ with property

$$[\hat{L}_z, \mathrm{e}^{i\varphi}] = \mathrm{e}^{i\varphi} \tag{43}$$

- Consider the hermitian combinations

$$\sin\varphi = \frac{1}{2i}\left(\mathrm{e}^{i\varphi} - \mathrm{e}^{-i\varphi}\right) , \quad \cos\varphi = \frac{1}{2}\left(\mathrm{e}^{i\varphi} + \mathrm{e}^{-i\varphi}\right) \tag{44}$$

- We have

$$[\hat{L}_z, \sin\varphi] = -i\cos\varphi\,, \quad [\hat{L}_z, \cos\varphi] = i\sin\varphi \tag{45}$$

- Introduce:

$$\Delta\hat{L}_z = \hat{L}_z - \langle L_z\rangle\,, \quad \Delta\sin\varphi = \sin\varphi - \langle\sin\varphi\rangle \tag{46}$$

- Use $[\Delta\hat{L}_z, \Delta\sin\varphi]$ and repeat the procedure for $\hat{x}$ and $\hat{p}$.

Intelligent states of $\hat{L}_z$ and $\sin\varphi$

- We are looking for states with the property that

$$\Delta L_z \,(\Delta \sin\varphi) = \tfrac{1}{2}|\langle \cos\varphi\rangle| \tag{47}$$

- A state $|\psi\rangle$ for which this holds must satisfy

$$\left(\hat{L}_z - \langle L_z\rangle\right)|\psi\rangle = i\alpha\,(\sin\varphi - \langle\sin\varphi\rangle)\,|\psi\rangle \tag{48}$$

- Using $\hat{L}_z \mapsto -id/d\varphi$, Eq.(48) becomes

$$\left(i\frac{d}{d\varphi} - \langle L_z\rangle\right)\psi(\varphi) = -i\alpha\,(\sin\theta - \langle\sin\theta\rangle)\,\psi(\varphi) \tag{49}$$

- As before, we have the restriction $\alpha < 0$.
- The solution is

$$\psi(\varphi) = \frac{1}{\sqrt{N}}\mathrm{e}^{i(\langle L_z\rangle\varphi - i\alpha\cos\varphi - i\alpha\varphi\langle\sin\varphi\rangle} \tag{50}$$

Intelligent states of $\hat{L}_z$ and $\sin\varphi$

- The condition $\psi(\varphi + 2\pi) = \psi(\varphi)$ implies

$$\langle L_z \rangle - i\langle \sin\varphi \rangle = m \,, \quad m \in \mathbb{Z} \,, \tag{51}$$

Thus

$$\psi(\varphi) = \frac{1}{\sqrt{N}} \mathrm{e}^{i(m\varphi - i\alpha\cos\varphi)} \tag{52}$$

- The normalization integral yields

$$\begin{aligned} 1 &= \frac{1}{N}\int_0^{2\pi} d\varphi\, \mathrm{e}^{2\alpha\cos\varphi} \,, && (53) \\ N &= 2\pi I_0(2\alpha) && (54) \end{aligned}$$

with I_n a modified Bessel function.
- Similarly,

$$\begin{aligned} \langle \cos\varphi \rangle &= \frac{1}{N}\int_0^{2\pi} d\varphi\, \mathrm{e}^{2\alpha\cos\varphi} \cos\varphi \,, && (55) \\ &= \frac{1}{N}\, 2\pi\, I_1(2\alpha) = \frac{I_1(2\alpha)}{I_0(2\alpha)} && (56) \end{aligned}$$

Summary

- Intelligent states are states with "saturate" the uncertainty relation:

$$\Delta A \Delta B = \tfrac{1}{2}|\langle[\hat{A},\hat{B}]\rangle| \tag{57}$$

- Intelligent states satisfy the eigenvalue equation

$$\left(\hat{A} - i\alpha\hat{B}\right)|\psi\rangle = \lambda|\psi\rangle \tag{58}$$

  with
  - $\lambda = \langle A\rangle - i\alpha\langle B\rangle$
  - $\alpha \in \mathbb{R}$,
  - $\alpha^2 = (\Delta A)^2 \,/\, (\Delta B)^2$

References

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