Intelligent states

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Intelligent states
Part 4: $SU(1,1)$ intelligence

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$su(1,1)$

The Lie algebra $su(1,1)$ occurs in models of parametric down-conversion. An input classical field α results in the production of two photons $a^\dagger b^\dagger$ or $a^\dagger a^\dagger$.

- The elements of the algebra are:

$$\begin{aligned} \hat{K}_x &= \tfrac{1}{4}\left(a^\dagger a^\dagger + aa\right), \; \hat{K}_y = \tfrac{1}{4i}\left(a^\dagger a^\dagger - aa\right), \\ \hat{K}_0 &= \tfrac{1}{4}\left(a^\dagger a + a a^\dagger\right) \end{aligned} \tag{1}$$

- They satisfy

$$[\hat{K}_x, \hat{K}_y] = -i\hat{K}_0\,, \quad [\hat{K}_y, \hat{K}_0] = i\hat{K}_x\,, \quad [\hat{K}_0, \hat{K}_x] = i\hat{K}_y\,. \tag{2}$$

- Intelligent states of $\hat{K}_x$ and $\hat{K}_y$ have been described as "minimum uncertainty states for amplitude–squared" fields.
- As always, define:

$$\hat{K}_\pm = \hat{K}_x \pm i\hat{K}_y, \tag{3}$$

$$\hat{K}_+ = \tfrac{1}{2}\,a^\dagger a^\dagger, \qquad \hat{K}_- = \tfrac{1}{2}\,a\,a. \tag{4}$$

The $\frac{1}{4}$ and $\frac{3}{4}$ series

- Let $|n\rangle$ be a harmonic oscillator state, with

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\,, \quad a|n\rangle = \sqrt{n}|n-1\rangle \tag{5}$$

- The state $|0\rangle$ satisfies

$$\hat{K}_0|0\rangle = \tfrac{1}{4}|0\rangle\,, \qquad \hat{K}_-|0\rangle = 0\,, \tag{6}$$

  so it is the bottom state of the $k = \frac{1}{4}$ series.
- In the same manner, $|1\rangle$ is the bottom state for the $k = \frac{3}{4}$ series.
- Thus, use $|k, m\rangle$ to denote states in the k series with

$$\hat{K}_0|k, m\rangle = m|k, m\rangle \tag{7}$$

- We have the correspondences

$$|0\rangle \mapsto |\tfrac{1}{4}, \tfrac{1}{4}\rangle\,, \qquad\qquad |1\rangle \mapsto |\tfrac{3}{4}, \tfrac{3}{4}\rangle\,. \tag{8}$$

- Since $\hat{K}_\pm$ cannot change the index k when acting on $|k, m\rangle$, we see the intelligent will belong to the $k = \frac{1}{4}$ series n or the $k = \frac{3}{4}$ series and will contain linear combos of even or odd $|n\rangle$

$SU(1,1)$ intelligence

- Intelligent states of $\hat{K}_x, \hat{K}_y$, are solutions to the eigenvalue equation

$$\left(\hat{K}_x - i\alpha\hat{K}_y\right)|\psi\rangle = \lambda|\psi\rangle\,, \qquad (9)$$

$$\tfrac{1}{2}\left((1-\alpha)\,\hat{K}_+ + (1+\alpha)\,\hat{K}_-\right)|\psi\rangle = \lambda|\psi\rangle \qquad (10)$$

- As $\hat{K}_\pm$ acts within an infinite dimensional space, there is an infinite number of solutions to the previous equation.

Associated Schrödinger problem

- Introduce the dummy variable ξ and the identifications

$$a^\dagger \longrightarrow \xi\,, \qquad a \longrightarrow \frac{d}{d\xi} \tag{11}$$

- Note that

$$[a^\dagger, a] = \mathbb{1} \quad \leftrightarrow \quad [\xi, \tfrac{d}{d\xi}] = \mathbb{1} \tag{12}$$

- Thus we can identify

$$\hat{K}_+ = \tfrac{1}{2} a^\dagger a^\dagger \longrightarrow \tfrac{1}{2}\xi^2\,, \quad \hat{K}_- = \tfrac{1}{2} a\, a \longrightarrow \tfrac{1}{2}\frac{d^2}{d\xi^2} \tag{13}$$

and

$$|n\rangle \longrightarrow \frac{\xi^n}{\sqrt{n!}}\,. \tag{14}$$

- The eigenvalue equation for intelligence becomes the differential equation

$$\tfrac{1}{4}\,(1+\alpha)\,\frac{d^2}{d\xi^2}\psi(\xi) + \tfrac{1}{4}\,(1-\alpha)\,\xi^2\psi(\xi) = \lambda\psi(\xi), \tag{15}$$

- This is the Schrödinger equation for the harmonic oscillator.

Associated Schrödinger problem

- The solutions are thus

$$\psi_n(\xi) \sim \mathrm{e}^{-\epsilon\,\xi^2/2} H_n(\sqrt{\epsilon}\,\xi), \tag{16}$$

$$= \sim \mathrm{e}^{-\frac{1}{2}\epsilon a^\dagger a^\dagger} H_n(\sqrt{\epsilon}\,a^\dagger)|0\rangle \tag{17}$$

$$= \mathrm{e}^{-\epsilon \hat{K}_+} H_n(\sqrt{\epsilon}\,a^\dagger)|0\rangle \tag{18}$$

  where

$$\epsilon = \sqrt{\frac{\alpha - 1}{\alpha + 1}}. \tag{19}$$

  and H_n is the n'th Hermite polynomial.
- For the solution to make sense as an operator equation, we need $|\epsilon| < 1$.
  - This implies $\alpha > 0$,
  - For $0 < \alpha < 1$, ϵ is purely imaginary,
  - For $1 < \alpha$, ϵ is real.

The case $0 \leq \alpha < 1$.

Assume $\alpha > 1$, so $\epsilon \in \mathbb{R}$.

- Define τ via

$$\alpha = \frac{1}{\cosh(\tau/2)}, \quad \Rightarrow \epsilon = i \tanh(\tau/2) \tag{20}$$

- Even or odd cases separately:
  - For $n = 2m$, $H_{2m}(\sqrt{\epsilon}\,\xi)$ is a polynomial of degree m in $\hat{K}_+$ acting on $|0\rangle$
  - for $n = 2m + 1$, $H_{2m+1}(\sqrt{\epsilon}\xi)$ is a polynomial of degree m in $\hat{K}_+$ acting on $|1\rangle$.

Partial disentanglement

To continue, start with

$$\psi_n(a^\dagger) = \mathrm{e}^{-\epsilon \hat{K}_+} H_n(\sqrt{\epsilon}\, a^\dagger)|0\rangle \tag{21}$$

- Observe that

$$\mathrm{e}^{-i\tau \hat{K}_x} = \mathrm{e}^{-\epsilon \hat{K}_+} \mathrm{e}^{\beta \hat{K}_0} \mathrm{e}^{-\epsilon \hat{K}_-}, \tag{22}$$

  where

$$\beta = -2 \ln\left(\cosh\left(\tau/2\right)\right). \tag{23}$$

- Multiply from the right by $\mathrm{e}^{\epsilon \hat{K}_-} \mathrm{e}^{-\beta \hat{K}_0}$:

$$\mathrm{e}^{-\epsilon \hat{K}_+} = \mathrm{e}^{-i\tau \hat{K}_x} \mathrm{e}^{\epsilon \hat{K}_-} \mathrm{e}^{-\beta \hat{K}_0}, \tag{24}$$

- Advantages:
  - $\hat{K}_0$ acts diagonally,
  - $\hat{K}_-$ is like a derivative that will act finitely many times since $H_n(\sqrt{\epsilon} a^\dagger)|0\rangle$ is like a polynomial of finite degree,
  - $\hat{K}_x$ is hermitian and $\mathrm{e}^{-i\tau K_x}$ can be calculated using known $SU(1,1)$ Wigner D-functions.

Partial disentanglement

- From:

$$\mathrm{e}^{-\beta \hat{K}_0} \left(\sqrt{\epsilon} a^\dagger\right)^p = \left(\sqrt{\epsilon} \cosh\left(\tau/2\right) a^\dagger\right)^p \, \mathrm{e}^{-\beta \hat{K}_0}$$

one shows

$$\mathrm{e}^{-\beta \hat{K}_0} H_n(\sqrt{\epsilon} a^\dagger)|0\rangle \sim H_n(\sqrt{\epsilon} \cosh(\tau/2) a^\dagger)|0\rangle \tag{25}$$

- To evaluate

$$\left(\mathrm{e}^{\frac{\epsilon}{2} a\, a}\right) H_n(\sqrt{\epsilon} \cosh(\tau/2) a^\dagger)|0\rangle. \tag{26}$$

  - Go back to

$$\mathrm{e}^{\frac{\epsilon}{2} \frac{d^2}{d\xi^2}} H_n(\sqrt{\epsilon} \cosh(\tau/2) \xi). \tag{27}$$

  - look at n even and odd separately,
  - use properties of Hermite polynomials.

The case of n even

Consider for simplicity the case where n even. Then:

- Expand the exponential:

$$e^{\frac{\epsilon}{2}\frac{d^2}{d\xi^2}} H_{2m}(\sqrt{\epsilon}\cosh(\tau/2)\,\xi)$$
$$= \sum_{p=0}^{m} \frac{1}{p!}\left(\frac{\epsilon}{2}\right)^p \left(\frac{d^{2p}}{d\xi^{2p}}\right) H_{2m}(\sqrt{\epsilon}\cosh(\tau/2)\xi). \quad (28)$$

- Express this as the series:

$$\begin{aligned} f(\xi) &= \mathrm{e}^{\frac{\epsilon}{2}\frac{d^2}{d\xi^2}} H_{2m}(\sqrt{\epsilon}\cosh(\tau/2)\,\xi) \\ &= c_0 + \frac{c_2}{2!}\xi^2 + \frac{c_4}{4!}\xi^4 + \ldots \\ &= \sum_{q=0}^{m} \frac{c_{2q}}{(2q)!}\xi^{2q}\,. \end{aligned} \quad (29)$$

c_0

- In particular,

$$c_0 = \sum_{p=0}^{m} \frac{1}{p!} (-1)^p \left(2\sinh^2(\tau/2)\right)^p \frac{(2m)!}{(2m-2p)!} H_{2m-2p}(0) \tag{30}$$

- Using

$$H_{2m-2p}(0) = (-1)^{m-p} \frac{(2m-2p)!}{(m-p)!} \tag{31}$$

we reduce this further to

$$\begin{aligned} c_0 &= (-1)^m \frac{(2m)!}{m!} \sum_{p=0}^{m} \left(2\sinh^2(\tau/2)\right)^p \frac{m!}{(m-p)!p!}, && (32) \\ &= (-1)^m \frac{(2m)!}{m!} \left(1 + 2\sinh^2(\tau/2)\right)^m, && (33) \\ &= (-1)^m \frac{(2m)!}{m!} (\cosh(\tau))^m. && (34) \end{aligned}$$

c_2 and c_{2q}

- After similar manipulations:

$$c_{2q} = (-1)^{m-q} \left(2i \sinh(\tau)\right)^q \frac{(2m)!}{(m-q)!} \left(\cosh(\tau)\right)^{m-q} . \quad (35)$$

- The solution

$$\psi_n(\xi) \sim \mathrm{e}^{-\epsilon\xi^2/2} H_n(\sqrt{\epsilon}\xi) \quad (36)$$

thus becomes

$$\begin{aligned} |\psi_n\rangle &= \mathrm{e}^{-i\tau\hat{K}_x}\, \mathrm{e}^{\epsilon\hat{K}_-} H_n(\sqrt{\epsilon}\cosh(\tau/2)\, a^\dagger)|0\rangle \\ &= \mathrm{e}^{-i\tau\hat{K}_x} \left(\sum_{m=0,2,\ldots,}^{n} c_m|m\rangle \right) , \\ &= \mathrm{e}^{-i\tau\hat{K}_x} \left(\sum_{r=0,1,\ldots}^{n/2} c_r|\tfrac{1}{4}, r+\tfrac{1}{4}\rangle \right) . \end{aligned} \quad (37)$$

Some examples

Write

$$|\psi_n\rangle = e^{-i\tau\hat{K}_x}\left(\sum_{r=0,2,\ldots,}^{n} c_r|r\rangle\right), \\ = e^{-i\tau\hat{K}_x}|\eta_n\rangle \tag{38}$$

for simplicity.
The first few $|\eta_n\rangle$s are:

$$|\eta_0\rangle = |0\rangle, \\ |\eta_2\rangle = \left(-\sqrt{2}\cosh(\tau)|0\rangle + 2i\sinh(\tau)|2\rangle\right). \tag{39}$$

Moreover,

$$\langle\tfrac{1}{4}, m|e^{-i\tau\hat{K}_x}|\tfrac{1}{4}, m'\rangle \sim \mathcal{D}^{1/4}_{mm'}(\pi/2, \tau, -\pi/2) \tag{40}$$

where $\mathcal{D}$ is an $SU(1,1)$ Wigner $\mathcal{D}$-function.

$SU(1,1)$ coupling

- We recall the product of two intelligent states is also intelligent.
- Write the associated Schrödinger problem for two independent intelligent states:

$$\frac{1}{4}(1+\alpha)\left(\frac{d^2}{d\xi_A^2}+\frac{d^2}{d\xi_B^2}\right)\psi(\xi_A,\xi_B) + \frac{1}{4}(1-\alpha)\left(\xi_A^2+\xi_B^2\right)\psi(\xi_A,\xi_B)=\lambda\psi(\xi_A,\xi_B) \quad (41)$$

- The resulting states are intelligent.

To understand the situation better:

- Write the wavefunction as a function of the operators:

$$\psi_{0,0}(\xi_A,\xi_B)\sim \mathrm{e}^{-\epsilon(\xi_A^2+\xi_B^2)/2}\longrightarrow \mathrm{e}^{-\epsilon(\hat{K}_{A,+}+\hat{K}_{B,+})}|0,0\rangle \quad (42)$$

- Now $|0,0\rangle$ is killed by $\hat{K}_-$ and satisfies

$$\hat{K}_0|0,0\rangle=\hat{K}_{0,A}|0\rangle_A|0\rangle_B+\hat{K}_{0,B}|0\rangle_A|0\rangle_B=\frac{1}{2}|0,0\rangle \quad (43)$$

$SU(1,1)$ coupling

- Thus,

$$\begin{aligned} \mathrm{e}^{-\epsilon(\hat{K}_{A,+}+\hat{K}_{B,+})}|0,0\rangle &\rightarrow \mathrm{e}^{-\epsilon(\hat{K}_{+})}|\tfrac{1}{2},\tfrac{1}{2}\rangle \\ &= |0,0\rangle - \epsilon\left(|1,0\rangle + |0,1\rangle\right) + \ldots \\ &= \sum_{m} c_{m=\frac{1}{2},\frac{3}{2}}(\epsilon)\,|\tfrac{1}{2},m\rangle \end{aligned} \tag{44}$$

- Note also $|1,0\rangle - |0,1\rangle$ is killed by $\hat{K}_-$ and is an eigenstate of $\hat{K}_0$ with eigenvalue $\frac{5}{2}$. Thus,

$$|\tfrac{5}{2},\tfrac{5}{2}\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle - |0,1\rangle\right) \tag{45}$$

- It is also possible to construct linear combinations of states $\sum_{p,q} c_{p,q}|p\rangle|q\rangle$ that are killed by $\hat{K}_-$ and are eigenstates of $\hat{K}_0$ with eigenvalue $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ etc.
- We conclude that, when putting together two systems of the $k=\frac{1}{4}$ series, we obtain the resulting values of $k=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$

Constructing higher k solutions by coupling

For k not $\frac{1}{4}$ or $\frac{3}{4}$, we have, in general:

$$\hat{K}_{\pm}|k,m\rangle = \sqrt{(m \pm k)(m \pm k + 1)}|k, m \pm 1\rangle \tag{46}$$

Using this and following the method of Lavoie for the coupling of angular momentum:

- If $|\psi_{n_A}\rangle$ and $|\psi_{n_B}\rangle$ are intelligent states of the $k = \frac{1}{4}$ families, then $|\psi_{n_A}\rangle|\psi_{n_B}\rangle$ is intelligent.
- Write

$$|\psi_{n_A}\rangle = \mathrm{e}^{-i\tau\hat{K}_{x,A}}|\eta_{n_A}\rangle \tag{47}$$

- Project by inserting the completeness relation for family k:

$$\begin{aligned} |\psi^{k}_{1/4,1/4}(\tau)\rangle &= \mathrm{e}^{-i\tau\hat{K}_x}\sum_m |k,m\rangle\langle\, k,m\,|\,\eta_{n_A};\eta_{n_B}\,\rangle\,, \\ &= \mathrm{e}^{-i\tau\hat{K}_x}\sum_m |k,m\rangle\kappa_m^{k,n_A,n_B} \end{aligned} \tag{48}$$

A recursion relation

We can evaluate $\langle\, km \,|\, \eta_{n_A}; \eta_{n_B} \,\rangle$ recursively:

- Start with

$$\begin{aligned} &\langle km | \mathrm{e}^{i\tau \hat{K}_x} \left(\hat{K}_x - i\alpha \hat{K}_y \right) [|\psi_{n_A}\rangle |\psi_{n_B}\rangle] \\ &\qquad = \lambda \langle km | \mathrm{e}^{i\tau \hat{K}_x} \mathrm{e}^{-i\tau \hat{K}_x} [|\eta_{n_A}\rangle |\eta_{n_B}\rangle] \\ &\qquad = \lambda \langle\, km \,|\, \eta_{n_A}; \eta_{n_B} \,\rangle \end{aligned} \tag{49}$$

- Next,

$$\begin{aligned} &\langle k, m | \mathrm{e}^{i\tau \hat{K}_x} \left(\hat{K}_x - i\alpha \hat{K}_y \right) \mathrm{e}^{-i\tau \hat{K}_x} [|\eta_{n_A}\rangle |\eta_{n_B}\rangle] \\ &= \langle k, m | \left(\hat{K}_x - i\alpha \cosh\tau \hat{K}_y - i\alpha \sinh\tau \hat{K}_z \right) [|\eta_{n_A}\rangle |\eta_{n_B}\rangle] \\ &= \langle k, m | \left(\hat{K}_- - i \tanh\tau \hat{K}_z \right) [|\eta_{n_A}\rangle |\eta_{n_B}\rangle] \end{aligned} \tag{50}$$

since $\alpha = 1/\cosh\tau$ and $\hat{K}_x - i\hat{K}_y = \hat{K}_-$.

A recursion relation

- Evaluating the matrix elements:

$$\begin{aligned}\lambda\langle\, km \,|\, \eta_{n_A}; \eta_{n_B}\,\rangle &= \langle k, m | \hat{K}_- \left[|\eta_{n_A}\rangle |\eta_{n_B}\rangle \right] - i \tanh\tau \langle k, m | \hat{K}_z \left[|\eta_{n_A}\rangle |\eta_{n_B}\rangle \right] \\ &= \sqrt{(k+m)(m-k+1)}\langle\, k, m+1 \,|\, \eta_{n_A}; \eta_{n_B}\,\rangle \\ &\quad -im \tanh\tau \langle\, k, m \,|\, \eta_{n_A}; \eta_{n_B}\,\rangle \end{aligned} \tag{51}$$

- Finally, we obtain

$$\langle\, k, m{+}1 \,|\, \eta_{n_A}; \eta_{n_B}\,\rangle = \frac{\lambda + im \tanh\tau}{\sqrt{(m+k)(m-k+1)}} \langle\, k, m \,|\, \eta_{n_A}; \eta_{n_B}\,\rangle \tag{52}$$

- This method allows use to generate all the coefficient from those for the lowest value $m = k$.

References

- G M Abd Al-Kader and A-S F Obada, Phys. Scr. **78** (2008) 035401
- Mark Hillery, Opt. Comm. **62** (1987) 135,
- Patrick Joanis, Dylan Mahler and Hubert de Guise, *SU(1,1) intelligent states and their squeezing properties*, in preparation.

Grand Summary

General summary:

- Intelligent states of $\hat{A}$ and $\hat{B}$ satisfy

$$\Delta A \Delta B = \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle|$$

- The intelligent states $|\psi\rangle$ are solutions to

$$\left(\hat{A} - i\alpha\hat{B}\right)|\psi\rangle = \lambda|\psi\rangle$$

  where α is real.
- Coherent states are intelligent, but not all intelligent states are coherent.
- Coherent states are never squeezed.
- Systematic construction method for:
  - $\hat{x}$ and $\hat{p}$,
  - $\hat{\varphi}$ and $\hat{L}_z$
  - $\hat{L}_x$ and $\hat{L}_y$
  - $\hat{K}_x$ and $\hat{K}_y$
- Because they saturate the uncertainty relation, they are a "natural" set to start an investigation of squeezing properties.

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Thunder Bay

Current Conditions

Observed at:	**Thunder Bay Airport**		
Date:	**10:00 AM EDT Thursday 16 April 2009**		
Condition:	**Mainly Sunny**	Temperature:	**5.2°C**
Pressure:	**103.2 kPa**	Dewpoint:	**-3.6°C**
Tendency:	**rising**	Humidity:	**53 %**
Visibility:	**32 km**	Wind:	**calm**

5 °C

Forecast

Today	Fri	Sat	Sun	Mon	Tue	Wed
19°C	17°C	4°C	5°C	9°C	9°C	8°C
	2°C	0°C	-5°C	-4°C	2°C	-3°C
	40%				60%	

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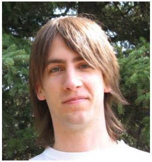
Thanks to:






From left to right:

- Ben Lavoie (now Ph.D at U. of Calgary),
- Ryan Vilim (now Ph.D at U. of Toronto),
- Patrick Joanis (now Ph.D at U. of Toronto),
- Dylan Mahler (now M.Sc. at U. of Toronto)
- Matt Milks (not shown)

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