Intelligent states

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Intelligent states Part 3: Squeezing

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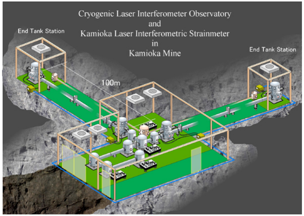
Overview

- Part 1: Introduction and simple cases
  - Basic definitions
  - Review of the uncertainty relation
  - Intelligence
  - Applications
- Part 2: Angular momentum intelligent states
  - Introduction and basic properties
  - A mathematical interlude
  - The method of Lavoie
  - The method of Rashid
- Part 3: Squeezing in angular momentum intelligent states.
  - x and p squeezing
  - What is spin squeezing
  - Some results and examples
- Part 4: $SU(1,1)$ intelligent states
  - An associated Schrödinger problem
  - $SU(1,1)$ coupling
  - Some results
- Summary

x and p squeezing: gravitational wave detection



- The current gravitational wave detectors are based on squeezing of quantized EM fields.
- Quantization of EM field results in harmonic oscillator–like Hamiltonian,
- Squeezing of EM field is same as harmonic oscillator squeezing in x and p.

Time evolution of Δx and Δp

- The harmonic oscillator squeezed states are not eigenstates of the Hamiltonian.
- Δx and Δp depend on time.

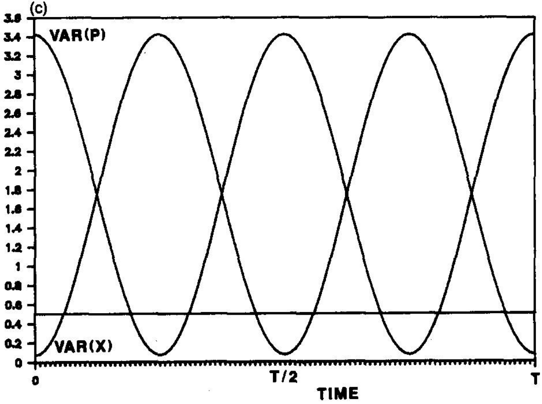


Figure: Δx and Δp as a function of time. From: Richard W. Henry and Sharon C. Glotzer, Am. J. Phys. **56** (1988) 318

Polarization squeezing

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LETTERS

Squeezing and over-squeezing of triphotons

L. K. Shalm[1], R. B. A. Adamson[1] & A. M. Steinberg[1]

Quantum mechanics places a fundamental limit on the accuracy of measurements. In most circumstances, the measurement uncertainty is distributed equally between pairs of complementary properties; this leads to the 'standard quantum limit' for measurement resolution. Using a technique known as 'squeezing', it is possible to reduce the uncertainty of one desired property below the standard quantum limit at the expense of increasing that of the complementary one. Squeezing is already being used to enhance the sensitivity of gravity-wave detectors[1] and may play a critical role in other high precision applications, such as atomic clocks[2] and optical communications[3]. Spin squeezing (the squeezing of angular momentum variables) is a powerful tool, particularly in the context of quantum light–matter interfaces[4–9]. Although impressive gains in squeezing have been made, optical spin-squeezed systems are still many orders of magnitude away from the maximum possible squeezing, known as the Heisenberg uncertainty limit. Here we demonstrate how an optical system can be squeezed essentially all the way to this fundamental bound. We construct spin-squeezed states by overlapping three indistinguishable photons in an optical fibre and manipulating their polarization (spin), resulting in the formation of a squeezed composite particle known as a 'triphoton'. The symmetry properties of polarization imply that the measured triphoton states can be most naturally represented by quasi-probability distributions on the surface of a sphere. In this work we show that the spherical topology of polarization imposes a limit on how much squeezing can occur, leading to the quasi-probability distributions wrapping around the sphere—a phenomenon we term 'over-squeezing'. Our observations of spin-squeezing in the few-photon regime could lead to new quantum resources for enhanced measurement, lithography and information processing that can be precisely engineered photon-by-photon.

The standard way to characterize and represent the polarization of classical light is through the use of a Stokes vector. The vector's components are the Stokes parameters S_1, S_2 and S_3 that describe the degree of linear, diagonal and circular polarization respectively, while the intensity of the beam is represented by the Stokes parameter S_0. S_1, S_2 and S_3 form a Cartesian coordinate system, and for polarized light $S_1^2 + S_2^2 + S_3^2 = S_0^2$. Such a Stokes vector terminates at a point on the surface of what is known as the Poincaré sphere. For quantum polarization states, a Poincaré sphere can be constructed where the Stokes parameters are replaced by the Stokes operators:

$$\begin{aligned}
\hat{S}_0 &= a_H^\dagger a_H + a_V^\dagger a_V = \hat{n}_H + \hat{n}_V = N, \\
\hat{S}_1 &= a_H^\dagger a_H - a_V^\dagger a_V = \hat{n}_H - \hat{n}_V, \\
\hat{S}_2 &= a_H^\dagger a_V + a_V^\dagger a_H = \hat{n}_D - \hat{n}_A, \\
\hat{S}_3 &= i(a_V^\dagger a_H - a_H^\dagger a_V) = \hat{n}_R - \hat{n}_L,
\end{aligned} \qquad (1)$$

where N is the total photon number and $a_i^\dagger$, a_i and $\hat{n}_i$ are the creation, annihilation and photon number operators for polarization mode

Squeezing: an operational definition

How to define squeezing.

- Given:
  - Two observables $\hat{A}$ and $\hat{B}$ with $[\hat{A}, \hat{B}] \neq 0$,
  - A reference state $|\psi\rangle_{\text{ref}}$,
  - Compute $(\Delta A)_{\text{ref}}$
  - Use this value as reference value, or "standard quantum limit".
- Given an arbitrary state $|\phi\rangle$:
  - Compute $(\Delta A)_{\phi}$.
  - If $(\Delta A)_{\phi} < (\Delta A)_{\text{ref}}$, $|\phi\rangle$ is squeezed.
- The concept is most useful when $|\psi\rangle_{\text{ref}}$ is experimentally or theoretically interesting.

Squeezing in harmonic oscillator states

- Take $|\psi\rangle_{\text{ref}} = |0\rangle$, the harmonic oscillator ground state.
- $(\Delta x)^2_{SQL} = (\Delta p)^2_{SQL} = \frac{1}{2}$ (in suitable units).
- Alternatively: $(\Delta x)^2_{SQL} = \frac{1}{2}|\langle[\hat{x}, \hat{p}]\rangle]|$ (in suitable units)
- Squeezing occurs when $(\Delta x)^2 < \frac{1}{2}|\langle[\hat{x}, \hat{p}]\rangle]|$.

Squeezing in harmonic oscillator states

- Consider

$$|\chi\rangle = \mathrm{e}^{i\chi(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a})}|0\rangle \tag{1}$$

- Using BCH, we get:

$$\mathrm{e}^{i\chi(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a})}\,\hat{x}\,\mathrm{e}^{i\chi(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a})} = \mathrm{e}^{\chi}\hat{x} \tag{2}$$

- Thus

$$\langle\chi|\hat{x}|\chi\rangle = \mathrm{e}^{\chi}\langle 0|\hat{x}|0\rangle = 0\,, \quad \langle\chi|\hat{x}^2|\chi\rangle = \tfrac{1}{2}\mathrm{e}^{2\chi}\,. \tag{3}$$

- Hence

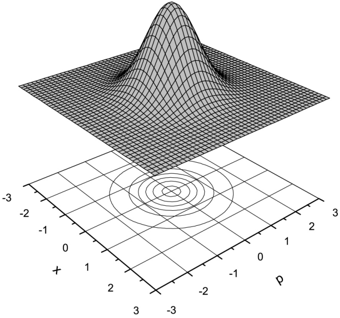
$$(\Delta x)^2_\chi = \tfrac{1}{2}\mathrm{e}^{2\chi} < \tfrac{1}{2}|\langle[\hat{x},\hat{p}]\rangle| \tag{4}$$

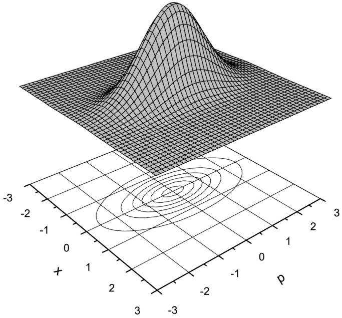
for any $\chi < 0$.

Illustrating HO squeezing with Wigner functions

- Illustrating squeezing is best done using functions of x and p.
- Wigner function

$$|\psi\rangle \mapsto W(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} d\xi \mathrm{e}^{-ip\xi}\psi^*(x-\tfrac{1}{2}\xi)\psi(x+\tfrac{1}{2}\xi) \quad (5)$$





Geometric properties

A simple translation of the state should not change its squeezing properties.

- Introduce the displacement operator

$$D(\alpha) = \mathrm{e}^{-i(\alpha^* \hat{a} + \alpha \hat{a}^\dagger)}\,, \qquad \alpha = \alpha_r + i\alpha_i \in \mathbb{C} \tag{6}$$

- The displacement shifts $\hat{x}$ and $\hat{p}$

$$D(\alpha)\,\hat{x}\,D^\dagger(\alpha) = \hat{x} + \alpha_r\,, \qquad D(\alpha)\,\hat{p}\,D^\dagger(\alpha) = \hat{p} + \alpha_i \tag{7}$$

  but does not affect Δx or Δp.

Geometric properties

Conclusion:

- $|0\rangle$ or $D(\alpha)|0\rangle$ are equally good reference states.

Define $|\alpha\rangle \equiv D(\alpha)|0\rangle$. Then:

- $(\Delta x)^2_\alpha = (\Delta x)^2_0 = \frac{1}{2}$.
- $|\alpha\rangle \equiv D(\alpha)|0\rangle$ is nothing but the usual harmonic oscillator coherent state.

For the coherent state $|\alpha\rangle$:

- $(\Delta x)^2_\alpha = \frac{1}{2}\,, (\Delta p)^2_\alpha = \frac{1}{2}$.
- $(\Delta x)(\Delta p) = \frac{1}{2}$
- Thus $|\alpha\rangle$ is intelligent

Squeezing in angular momentum states

Definition:

- Select $|\ell, \ell\rangle$ as reference state.
- Introduce the "displacements on the sphere" operator

$$R(\alpha, \beta, \gamma) = \mathrm{e}^{-i\varphi \hat{L}_z}\, \mathrm{e}^{-i\beta \hat{L}_y} \mathrm{e}^{-i\gamma \hat{L}_z} \tag{8}$$

  and a set of equivalent reference states:

$$\begin{aligned} R(\alpha, \beta, \gamma)|\ell, \ell\rangle &\sim \mathrm{e}^{-i\varphi \hat{L}_z}\, \mathrm{e}^{-i\beta \hat{L}_y} |\ell, \ell\rangle \\ &\equiv |\varphi, \beta\rangle \,. \end{aligned} \tag{9}$$

- $|\varphi, \beta\rangle$ is an angular momentum coherent state.
- $|0, \beta\rangle$ are the intelligent states of Lecture 2.

Illustrating the coherent state

To illustrate the effect of the rotation, use SU(2) Wigner function:

- Start with the Wigner kernel:

$$\hat{w}(\theta,\varphi) = \sqrt{\frac{4\pi}{2\ell+1}} \sum_{L=0}^{2\ell} \sum_{-M}^{M} Y^{*}_{L,M}(\theta,\varphi)\, \hat{T}^{\ell}_{L,M} \qquad (10)$$

- Here,

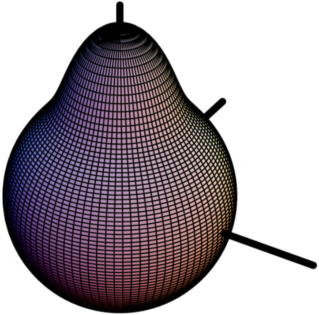
$$\hat{T}^{\ell}_{L,M} = \sqrt{\frac{2L+1}{2\ell+1}} \sum_{m,m'=\ell}^{\ell} C^{\ell,m'}_{\ell,m;L,M} |\ell, m'\rangle\langle \ell, m| \qquad (11)$$

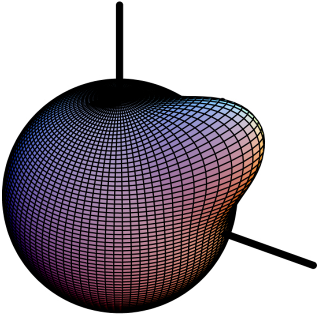
  are tensor operators.
- The Wigner function is just the trace of the Wigner kernel over the state:

$$W^{\ell}_{\ell_a,\ell_b}(\beta;\theta,\varphi) = \mathrm{Tr}\left(\hat{w}|\psi^{\ell}_{\ell_a,\ell_b}(\beta)\rangle\langle\psi^{\ell}_{\ell_a,\ell_b}(\beta)|\right) \qquad (12)$$

Illustrating the coherent state

A "displacement on the sphere" does not induce any squeezing of the state.





Defining squeezing on the sphere

Naive definition:

- Choose $|\ell,\ell\rangle$ as reference state.
- Then

$$(\Delta L_x)^2 = \tfrac{1}{2}|\langle \hat{L}_z\rangle| = \tfrac{1}{2}\ell, \quad (\Delta L_y)^2 = \tfrac{1}{2}|\langle \hat{L}_z\rangle| = \tfrac{1}{2}\ell \qquad (13)$$

- Squeezing occurs when $(\Delta L_x)^2 = \tfrac{1}{2}|\langle \hat{L}_z\rangle|$.

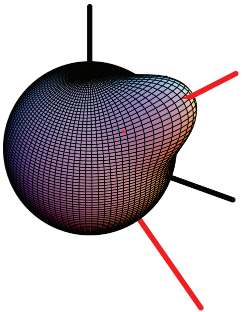
Difficulties:

- Choose an equivalent reference state like $|\beta\rangle = \mathrm{e}^{-i\beta\hat{L}_y}|\ell,\ell\rangle$
- This state is just a rotated version of $|\ell,\ell\rangle$ but...
- $(\Delta L_x)^2 < \tfrac{1}{2}\ell$.

Contrasts:

- With $\hat{x},\hat{p}$, a translation in the plane $\hat{x}\longrightarrow\hat{x}+\alpha_i$ with the constant α_i not affecting Δx.
- With $\hat{L}_x$, a translation on the sphere $\hat{L}_x\longrightarrow\cos\beta\hat{L}_x-\sin\beta\hat{L}_y$, not a constant.

Defining squeezing on the sphere



- Define $\hat{z}'$ as the direction of $(\langle \hat{L}_x \rangle, \langle \hat{L}_y \rangle, \langle \hat{L}_z \rangle)$
- $\hat{L}_{z'} = \mathrm{e}^{i\beta \hat{L}_y} \hat{L}_z \mathrm{e}^{-i\beta \hat{L}_y}$ is along $\hat{z}'$.
- $\hat{L}_{x'} = \mathrm{e}^{i\beta \hat{L}_y} \hat{L}_x \mathrm{e}^{-i\beta \hat{L}_y}$ is along $\hat{x}'$.
- $(\Delta L_{x'})^2_{SQL} = \frac{1}{2} |\langle [\hat{L}_{x'}, \hat{L}_{y'}] \rangle|$.
- This makes $(\Delta L_{x'})^2 < (\Delta L_{x'})^2_{SQL}$ independent of rotations on the sphere.

Defining squeezing on the sphere

Issues with the use of Wigner functions:

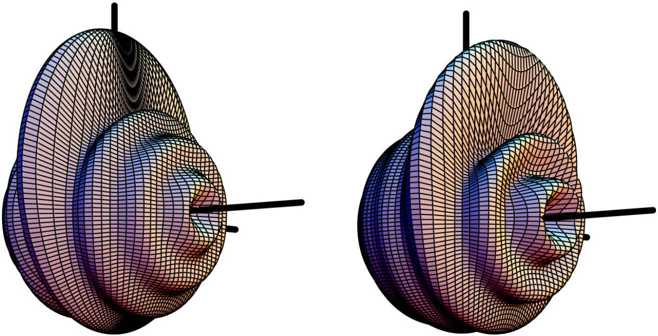


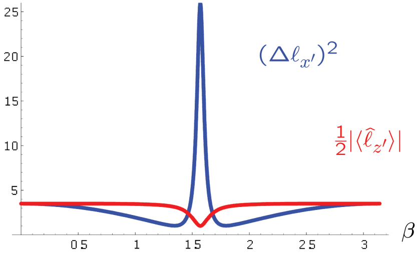
Figure: Left: squeezed state. Right: not squeezed

Squeezing plots

- Select a (normalized) angular momentum intelligent state

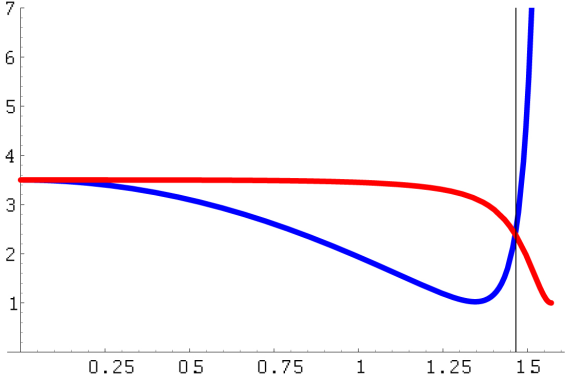
$$|\psi^{\ell}_{\ell_A,\ell_B}(\beta)\rangle = \sum_m \kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)|\ell, m\rangle \tag{14}$$

- Compute $(\Delta L_{x'})^2$ and $\frac{1}{2}|\langle \hat{L}_{z'}\rangle|$.
- Using $\ell_A = 9/2, \ell_B = 5/2, \ell = 7$, we obtain:



Squeezing plots

For $\ell_A = 9/2, \ell_B = 5/2, \ell = 7$:



- There is some squeezing in $|\psi^{7}_{9/2,5/2}(\beta)\rangle$ for the range $0 < \beta \leq 1.43$.

Trends

For angular momentum intelligent states $|\psi^{\ell}_{\ell_A,\ell_B}(\alpha)\rangle$:

- For $\ell_B = 0$, we have $\ell = \ell_A$. The state $|\psi^{\ell}_{\ell,0}(\alpha)\rangle$ is a coherent state and by definition never squeezed.
- For $\ell_B \neq 0$, there is always some range of β for which there is squeezing, and a range for which there is no squeezing.
  - The range of β for which there is squeezing increases with increasing ℓ_B and decreasing $\ell_A = \ell - \ell_B$ *until* $\ell_B = \ell_A$ or $\ell_B = \ell_A - \frac{1}{2}$.
  - With $\ell_B = \ell_A$, squeezing occurs for every value of β. The state $|\psi^{\ell}_{\ell/2,\ell_2}(\beta)\rangle$ is "anticoherent".
  - When $\ell_B > \ell_A$, the results are those of the $\ell_A > \ell_B$ case provided $\ell_B \to \ell_A$, $\ell_A \to \ell_B$.

The angular momentum coherent states

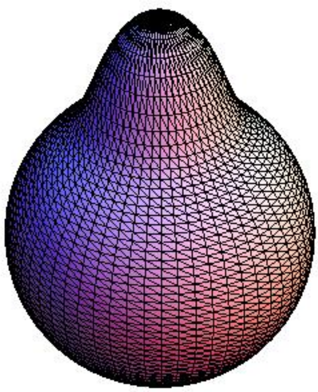
Main features:

- By definition, coherent states are never squeezed.
- Rotating a coherent state will not deform the Wigner function.
- Recall

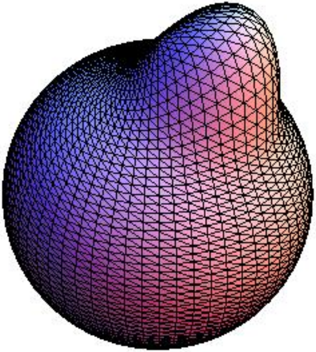
$$|\beta\rangle = \mathrm{e}^{-i\beta\hat{L}_y}|\ell,\ell\rangle \tag{15}$$

- With $\beta = 0$, we get a "North Pole" state.

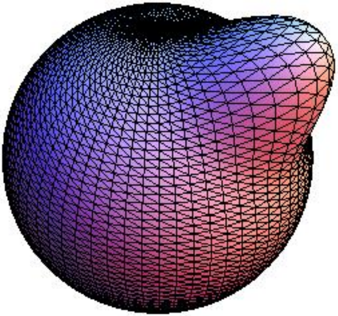
Coherent: $\ell = 9, \beta = 0$



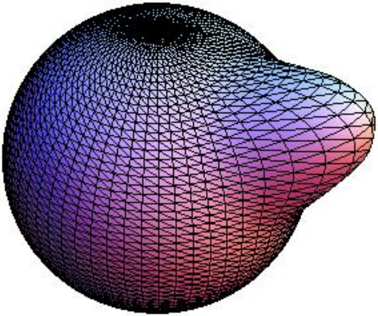
Coherent: $\ell = 9, \beta = \pi/9$



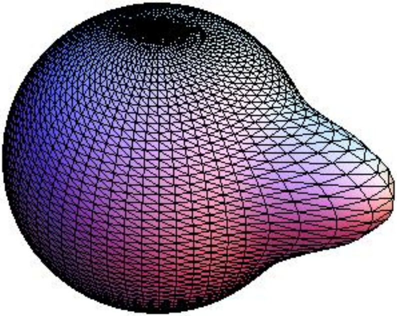
Coherent: $\ell = 9, \beta = 2\pi/9$



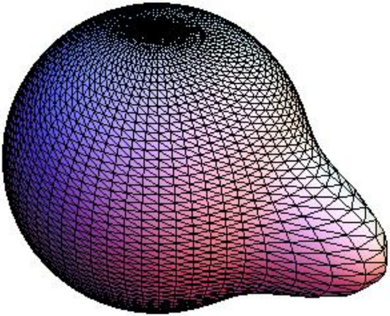
Coherent: $\ell = 9, \beta = 3\pi/9$



Coherent: $\ell = 9, \beta = 4\pi/9$



Coherent: $\ell = 9, \beta = 5\pi/9$



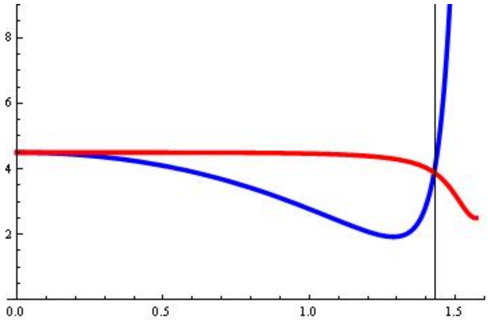
Intelligent state with $\ell_A = 7, \ell_B = 2$

Main features:

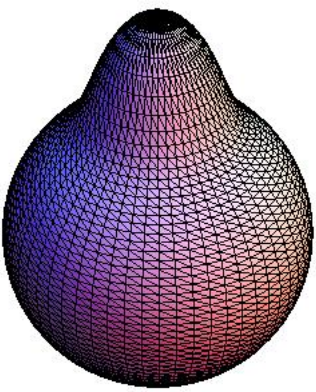
- $|\psi^9_{7,2}(\alpha)\rangle$ is obtained from

$$\sum_M |9, M\rangle\langle 9, M| \left[\mathrm{e}^{-i\beta\hat{L}_y}|7,7\rangle\right]\left[\mathrm{e}^{i\beta\hat{L}_y}|2,2\rangle\right] \qquad (16)$$

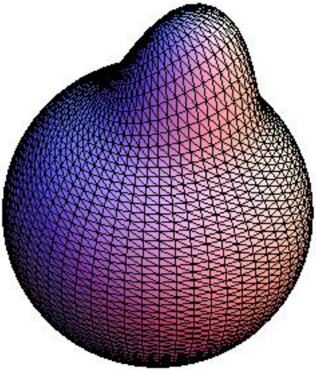
- For $\beta = 0$, we get a "North Pole" state,
- Squeezing occurs for $0 < \beta \leq 1.43$



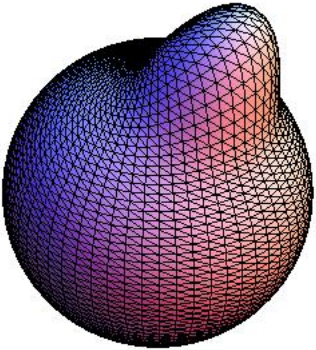
Intelligent: $\ell_A = 7, \ell_B = 2, \beta = 0.$



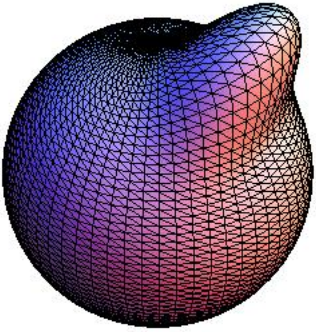
Intelligent: $\ell_A = 7, \ell_B = 2, \beta = \pi/9$



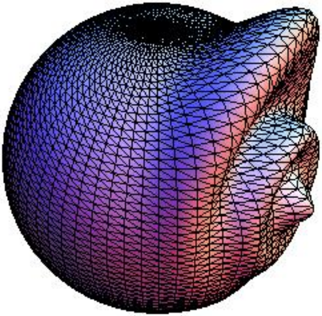
Intelligent: $\ell_A = 7, \ell_B = 2, \beta = 2\pi/9$



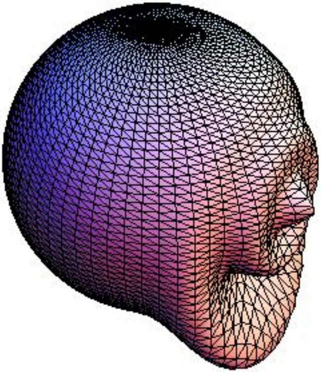
Intelligent: $\ell_A = 7, \ell_B = 2, \beta = 3\pi/9$



Intelligent: $\ell_A = 7, \ell_B = 2, \beta = 4\pi/9$



Intelligent: $\ell_A = 7, \ell_B = 2, \beta = 5\pi/9$



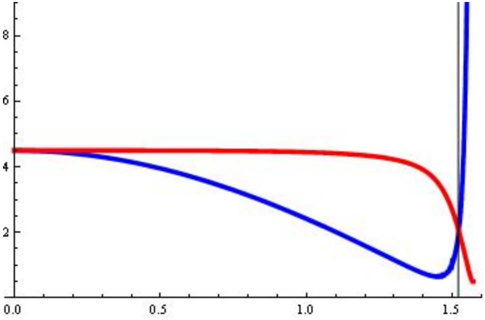
Intelligent state with $\ell_A = 5, \ell_B = 4$

Main features:

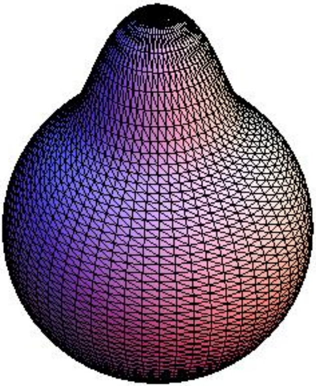
- $|\psi^{9}_{5,4}(\alpha)\rangle$ is obtained from

$$\sum_M |9, M\rangle\langle 9, M| \left[\mathrm{e}^{-i\beta\hat{L}_y}|5,5\rangle\right]\left[\mathrm{e}^{i\beta\hat{L}_y}|4,4\rangle\right] \tag{17}$$

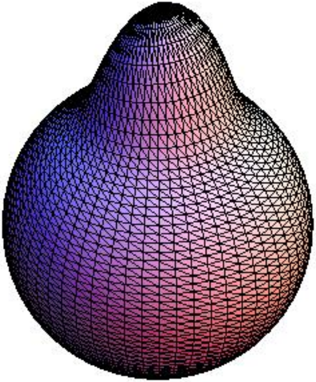
- For $\beta = 0$, we get a "North Pole" state,
- Squeezing occurs for $0 < \beta \leq 1.52$



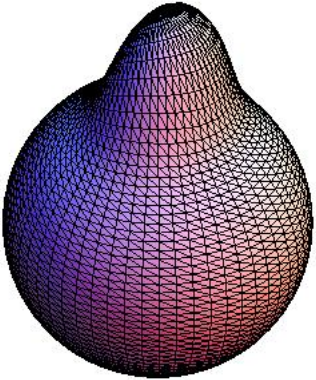
Intelligent: $\ell_A = 5, \ell_B = 4, \beta = 0.$



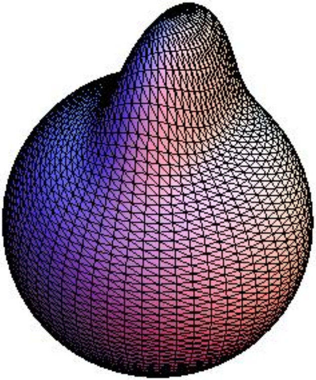
Intelligent: $\ell_A = 5, \ell_B = 4, \beta = \pi/9$



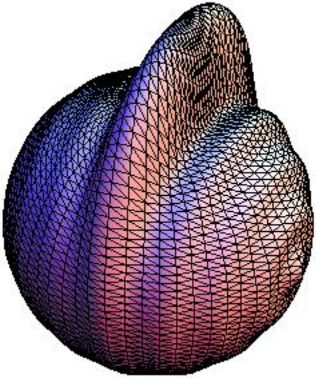
Intelligent: $\ell_A = 5, \ell_B = 4, \beta = 2\pi/9$



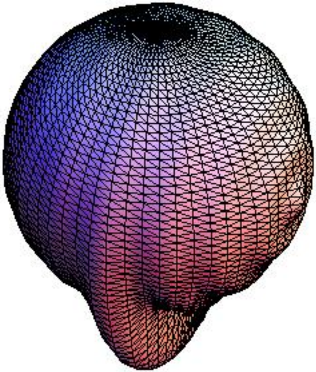
Intelligent: $\ell_A = 5, \ell_B = 4, \beta = 3\pi/9$



Intelligent: $\ell_A = 5, \ell_B = 4, \beta = 4\pi/9$



Intelligent: $\ell_A = 5, \ell_B = 4, \beta = 5\pi/9$



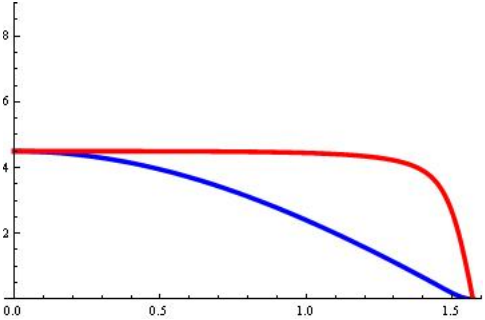
Intelligent state with $\ell_A = 9/2, \ell_B = 9/2$

Main features:

- $|\psi^9_{9/2,9/2}(\alpha)\rangle$ is obtained from

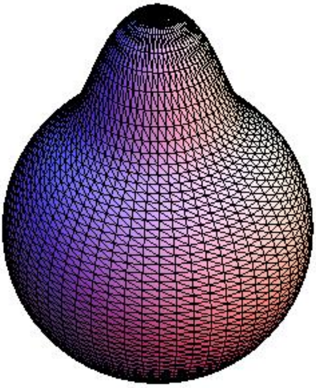
$$\sum_M |9, M\rangle\langle 9, M| \left[\mathrm{e}^{-i\beta\hat{L}_y}|\tfrac{9}{2}, \tfrac{9}{2}\rangle\right] \left[\mathrm{e}^{i\beta\hat{L}_y}|\tfrac{9}{2}, \tfrac{9}{2}\rangle\right] \tag{18}$$

- For $\beta = 0$, we get a "North Pole" state,
- Squeezing occurs for every β

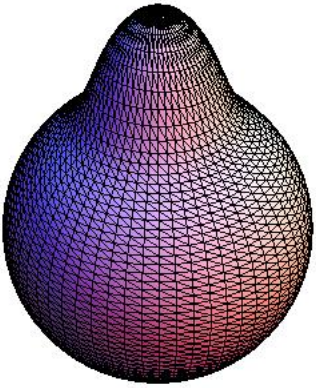


- The state is "anti–coherent".

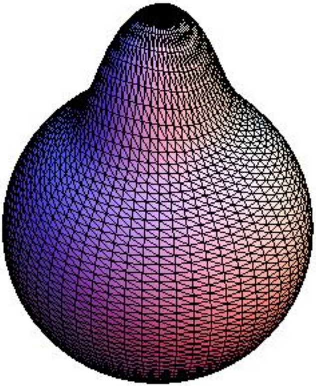
Intelligent: $\ell_A = 9/2, \ell_B = 9/2, \beta = 0.$



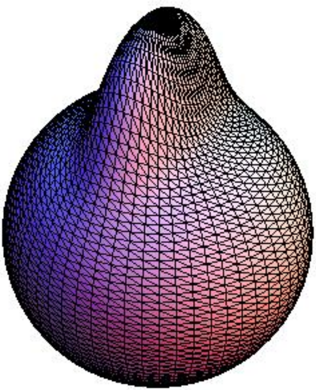
Intelligent: $\ell_A = 9/2, \ell_B = 9/2, \beta = \pi/9$



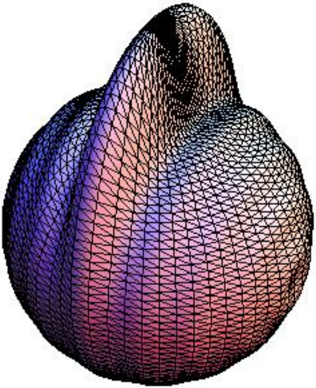
Intelligent: $\ell_A = 9/2, \ell_B = 9/2, \beta = 2\pi/9$



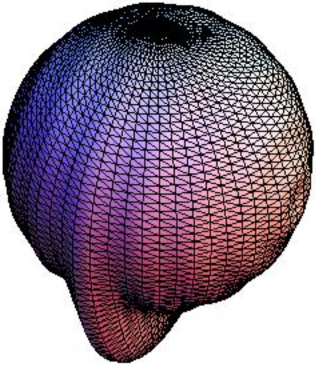
Intelligent: $\ell_A = 9/2, \ell_B = 9/2, \beta = 3\pi/9$



Intelligent: $\ell_A = 9/2, \ell_B = 9/2, \beta = 4\pi/9$

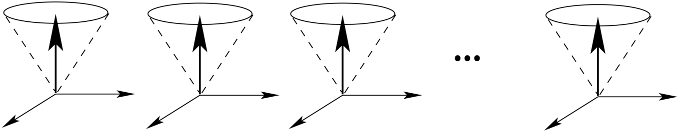


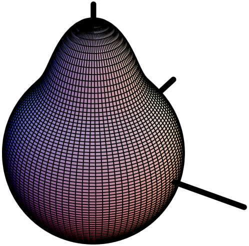
Intelligent: $\ell_A = 9/2, \ell_B = 9/2, \beta = 5\pi/9$

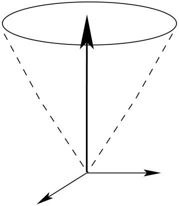


A physical model?

For a coherent state like $|\ell_A, \ell_A\rangle$:



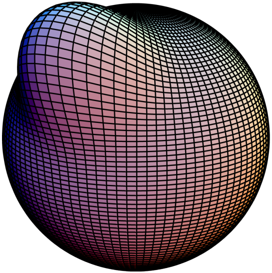


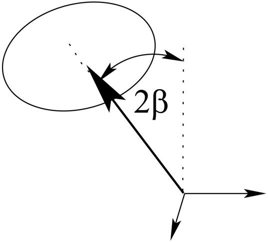


- In the plane perpendicular to $\langle L_{z',A}\rangle$, the components $L_{y',A}$ and $L_{x',A}$ are randomly distributed.
- $\langle \hat{L}_{x'} \rangle = \langle \hat{L}_{y'} \rangle = 0$.

A physical model?

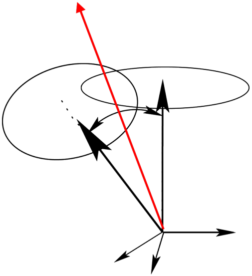
Put this together with a rotated coherent state:





A physical model?

The result is



- The projections of each angular momentum $L_{z,A}$ and $L_{z,B}$ in the plane perpendicular to their respective axes are circles.
- The projections of each angular momentum $L_{z,A}$ and $L_{z,B}$ in the plane perpendicular to the average L_z is are now ellipses, *i.e.* squeezed circles.
- The vectors $\hat{L}_{x,A}$, $\hat{L}_{x,B}$ $\hat{L}_{y,A}$, $\hat{L}_{y,B}$ are no longer randomly distributed on in the plane perpendicular to $\langle L_z \rangle$.
- Explains why there is squeezing for small β, but not why squeezing disappears near $\beta \approx \pi/2$.

Summary

Summary:

- Coherent states are usual reference states to establish a standard quantum limit. Thus, coherent states are never squeezed.
- Because coherent states are "displacements" of some reference ket, a careful analysis is required to define squeezing so it does not depend on such displacements.
  - One must establish a local vertical and a resulting system of axes using the vector $\langle \vec{L} \rangle$.
  - One must measure average values and fluctuations of observables defined relative to this new set of axes.
- Every angular momentum intelligent state (except the angular momentum coherent state) has a regime over which there is squeezing.
- For angular momentum intelligent states, squeezing is difficult to visualize using Wigner distributions.
- A simple but comprehensive model of squeezing remains elusive.

References

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