Intelligent states

Dr. Hubert de Guise
Department of Physics
Lakehead University
Thunder Bay, Ontario, Canada

hubert.deguise@lakeheadu.ca

Intelligent states Part 2: Angular momentum intelligent states

Dr. Hubert de Guise

Department of Physics, Lakehead University

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$su(2)$: the angular momentum algebra

- The angular momentum algebra contains three elements: $\hat{L}_x, \hat{L}_y$ and $\hat{L}_z$.
- They have non–zero commutation relations

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hat{L}_z\,, \quad \left[\hat{L}_y, \hat{L}_z\right] = i\hat{L}_x \quad \left[\hat{L}_z, \hat{L}_x\right] = i\hat{L}_y \tag{1}$$

- It is convenient to introduce (non-hermitian) raising and lowering operators:

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y\,, \quad \hat{L}_- = \hat{L}_x - i\hat{L}_y \tag{2}$$

- We now have the non–zero commutation relations

$$\left[\hat{L}_z, \hat{L}_\pm\right] = \pm\hat{L}_\pm\,, \; \left[\hat{L}_+, \hat{L}_-\right] = 2\hat{L}_0\,. \tag{3}$$

- We have the actions

$$\begin{aligned} \hat{L}_z|\ell, m\rangle &= m|\ell, m\rangle\,, \\ \hat{L}_\pm|\ell, m\rangle &= \sqrt{(\ell \mp m)(\ell \pm m + 1)}|\ell, m \pm 1\rangle \end{aligned} \tag{4}$$

Physical realization

- Let $a_1, a_2, a_1^\dagger, a_2^\dagger$ denote the usual creation and destruction operators.
- We have

$$\begin{aligned}
a_1^\dagger a_2 |n_1, n_2\rangle &= \sqrt{(n_1+1)n_2}|n_1+1, n_2-1\rangle\,, \\
a_2^\dagger a_1 |n_1, n_2\rangle &= \sqrt{n_1(n_2+1)}|n_1-1, n_2+1\rangle\,, \\
\tfrac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right)|n_1, n_2\rangle &= \tfrac{1}{2}(n_1-n_2)|n_1, n_2\rangle \qquad (5)
\end{aligned}$$

- Now look at

$$\begin{aligned}
\left[\tfrac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right), a_1^\dagger a_2\right] &= a_1^\dagger a_2, \\
\left[\tfrac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right), a_2^\dagger a_1\right] &= -a_2^\dagger a_1, \\
\left[a_1^\dagger a_2, a_2^\dagger a_1\right] &= 2 \times \tfrac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right) \qquad (6)
\end{aligned}$$

- Those are the commutation relations of $\hat{L}_\pm, \hat{L}_z$.

$su(2)$ in quantum optics

Thus:

- We have the correspondence between operators

$$\hat{L}_z \to \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right)\,, \quad \hat{L}_+ = a_1^\dagger a_2\,, \quad \hat{L}_- = a_2^\dagger a_1 \tag{7}$$

- We have the correspondence between angular momentum and number labels:

$$j = \tfrac{1}{2}(n_1+n_2)\,, m = \tfrac{1}{2}(n_1-n_2), \quad n_1 = j+m\,, n_2 = j-m \tag{8}$$

- These correspondences is central in quantum optics:
  - If the indices 1 and 2 denote arms of an interferometer, we can use $su(2)$ to describe passive lossless interferometers.
  - if the indices 1 and 2 denote horizontal or vertical polarization, we can describe polarization of light.

Review and setup for intelligent states

- Intelligent states of $\hat{x}$ and $\hat{p}$ satisfy $\Delta x \Delta p = \frac{1}{2}$.
- They are solutions to

$$(\hat{x} - i\alpha\hat{p})\,|\chi\rangle = \lambda|\chi\rangle \tag{9}$$

- Set $\alpha = -1$: $\hat{x} + i\hat{p} = a$. The solution to

$$a|\chi\rangle = \lambda|\chi\rangle \tag{10}$$

  is the harmonic oscillator coherent state of quantum optics.
- Intelligent states $\left|\psi^{\ell}\left(\alpha\right)\right\rangle$ are states that satisfy ($\hbar = 1$)

$$\Delta L_x \Delta L_y = \tfrac{1}{2}|\langle \hat{L}_z\rangle|. \tag{11}$$

- They are solution to

$$(\hat{L}_x - i\alpha\hat{L}_y)|\psi^{\ell}(\alpha)\rangle = \lambda|\psi^{\ell}(\alpha)\rangle \tag{12}$$

- Set $\alpha = \mp 1$: $\hat{L}_x \pm i\hat{L}_y = \hat{L}_{\pm}$. The solutions to

$$L_{\pm}|\psi^{\ell}\rangle = \lambda|\psi^{\ell}\rangle \tag{13}$$

  are the angular momentum kets $|\ell, \pm\ell\rangle$. These imply $\lambda = 0$.

Basic properties of angular momentum intelligent states

- Since $\hat{L}_x$ and $\hat{L}_y$ act in a finite dimensional space and are hermitian:
  - The minimum in the product $\Delta L_x \Delta L_y$ is 0,
  - This minimum is reached by using an eigenstate of either $\hat{L}_x$ or $\hat{L}_y$.
  - intelligent states are **not** minimum uncertainty states.

- Intelligent states $\left|\psi^{\ell}(\alpha)\right\rangle$ satisfy (as always) the eigenvalue equation

$$(\hat{L}_x - i\alpha\hat{L}_y)|\psi^{\ell}(\alpha)\rangle = \lambda\,|\psi^{\ell}(\alpha)\rangle, \tag{14}$$

  for the non–hermitian operators $\hat{L}_x - i\alpha\hat{L}_y$,
- $-\infty \leq \alpha \leq \infty$ is a real parameter.
- The eigenvalue λ is related to the average value of $\hat{L}_x$ and $\hat{L}_y$ and to the parameter α via:

$$\lambda = \langle\, L_x\rangle - i\alpha\langle\, L_y\rangle\,. \tag{15}$$

BCH and angular momentum operators

- Recall Baker-Campbell-Hausdorff:

$$\mathrm{e}^{\hat{A}}\,\hat{B}\,\mathrm{e}^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \tfrac{1}{2}[\hat{A},[\hat{A},\hat{B}]] + \dots \tag{16}$$

- Apply this to $\mathrm{e}^{i\beta\hat{L}_y}\,\hat{L}_x\,\mathrm{e}^{-i\beta\hat{L}_y}$:

$$\begin{aligned}\left[i\beta\hat{L}_y,\hat{L}_x\right] = i\beta(-i)\hat{L}_z &= \beta L_z\\ \left[i\beta\hat{L}_y,\left[i\beta\hat{L}_y,\hat{L}_x\right]\right] = \left[i\beta\hat{L}_y,\beta L_z\right] &= i\beta^2(i)\hat{L}_x = -\beta^2\hat{L}_x\end{aligned}$$

- Continue but we can already guess the result:

$$\begin{aligned}\mathrm{e}^{i\beta\hat{L}_y}\,\hat{L}_x\,\mathrm{e}^{-i\beta\hat{L}_y} &= \hat{L}_x\left(1-\tfrac{1}{2}\beta^2+\dots\right)+\hat{L}_z\left(\beta+\dots\right)\\ &= \hat{L}_x\cos\beta+\hat{L}_z\sin\beta \qquad (17)\end{aligned}$$

- Similarly

$$\begin{aligned}\mathrm{e}^{i\beta\hat{L}_y}\,\hat{L}_z\,\mathrm{e}^{-i\beta\hat{L}_y} &= \hat{L}_z\cos\beta-\hat{L}_x\sin\beta\,, \qquad (18)\\ \mathrm{e}^{i\beta\hat{L}_y}\,\hat{L}_y\,\mathrm{e}^{-i\beta\hat{L}_y} &= \hat{L}_y \qquad (19)\end{aligned}$$

Some angular momentum coherent states

Start with the intelligent state $|\ell, \ell\rangle$.

- Write
$$|\beta\rangle = \mathrm{e}^{-i\beta\hat{L}_y}|\ell, \ell\rangle\,. \tag{20}$$
- $|\beta\rangle$ is a special case of the more general angular momentum coherent states
$$|\gamma, \beta\rangle \equiv \mathrm{e}^{-i\gamma\hat{L}_z}\,\mathrm{e}^{-i\beta\hat{L}_y}|\ell, \ell\rangle\,. \tag{21}$$
- Note that
$$\begin{aligned}\langle\beta|\hat{L}_z|\beta\rangle &= \langle\ell,\ell|\mathrm{e}^{i\beta\hat{L}_y}\hat{L}_z\mathrm{e}^{-i\beta\hat{L}_y}|\ell,\ell\rangle\,, \\ &= \cos\beta\langle\ell,\ell|\hat{L}_z|\ell,\ell\rangle - \sin\beta\langle\ell,\ell|\hat{L}_x|\ell,\ell\rangle\,, \\ &= \ell\cos\beta\,. \end{aligned}\tag{22}$$

Some angular momentum coherent states

Also:
$$\begin{aligned}\langle\beta|\hat{L}_y|\beta\rangle &= \langle\ell,\ell|\mathrm{e}^{i\beta\hat{L}_y}\hat{L}_y\mathrm{e}^{-i\beta\hat{L}_y}|\ell,\ell\rangle\,,\\ &= \langle\ell,\ell|\hat{L}_y|\ell,\ell\rangle = 0\,,\end{aligned}\tag{23}$$

$$\langle\beta|(\hat{L}_y)^2|\beta\rangle = \langle\ell,\ell|(\hat{L}_y)^2|\ell,\ell\rangle = \tfrac{1}{2}\ell\,,\tag{24}$$
$$(\Delta L_y)^2 = \tfrac{1}{2}\ell\tag{25}$$

while
$$\begin{aligned}\langle\beta|\hat{L}_x|\beta\rangle &= \langle\ell,\ell|\mathrm{e}^{i\beta\hat{L}_y}\hat{L}_x\mathrm{e}^{-i\beta\hat{L}_y}|\ell,\ell\rangle\,,\\ &= \cos\beta\langle\ell,\ell|\hat{L}_x|\ell,\ell\rangle + \sin\beta\langle\ell,\ell|\hat{L}_z|\ell,\ell\rangle\,,\\ &= \ell\sin\beta\end{aligned}\tag{26}$$

and
$$\begin{aligned}\langle\beta|(\hat{L}_x)^2|\beta\rangle &= \cos^2\beta\langle\ell,\ell|(\hat{L}_x)^2|\ell,\ell\rangle + \sin^2\beta\langle\ell|(\hat{L}_z)^2|\ell\rangle\\ &\quad+\sin\beta\cos\beta\langle\ell,\ell|(\hat{L}_x\hat{L}_z+\hat{L}_z\hat{L}_x)|\ell,\ell\rangle\,,\\ &= \tfrac{1}{2}\ell\cos^2\beta + \ell^2\sin^2\beta\,,\end{aligned}\tag{27}$$

Hence $(\Delta L_x)^2 = \frac{1}{2}\ell\cos^2\beta\,.$

Some angular momentum coherent states

- Collecting the results:

$$(\Delta L_y)^2(\Delta L_x)^2 = \tfrac{1}{4}\ell^2\cos\beta = \tfrac{1}{4}|\langle \hat{L}_z\rangle|^2\,. \tag{28}$$

- Conclusion: the coherent states $|\beta\rangle = \mathrm{e}^{-i\beta\hat{L}_y}|\ell,\ell\rangle$ are intelligent.
- Alternatively:

$$\begin{aligned}\left(\hat{L}_x - i\alpha\hat{L}_y\right)\mathrm{e}^{-i\beta\hat{L}_y}|\ell,\ell\rangle &= \lambda\mathrm{e}^{-i\beta\hat{L}_y}|\ell,\ell\rangle\,,\\ &= \mathrm{e}^{-i\beta\hat{L}_y}\,\mathrm{e}^{i\beta\hat{L}_y}\left(\hat{L}_x - i\alpha\hat{L}_y\right)\mathrm{e}^{-i\beta\hat{L}_y}|\ell,\ell\rangle\end{aligned} \tag{29}$$

- which implies

$$\mathrm{e}^{-i\beta\hat{L}_y}\left(\hat{L}_z\sin\beta + [\hat{L}_x\cos\beta - i\alpha\hat{L}_y]\right)|\ell,\ell\rangle \tag{30}$$

- Choose $\alpha = -\cos\beta$ so $\hat{L}_x\cos\beta - i\alpha\hat{L}_y = \hat{L}_+$. The eigenvalue is $\lambda = \ell\sin\beta$.

Coupling property

Because the eigenvalue equation

$$(\hat{L}_x - i\alpha\hat{L}_y)|\psi^\ell(\alpha)\rangle = \lambda\,|\psi^\ell(\alpha)\rangle, \tag{31}$$

is *linear* in $\hat{L}_x - i\alpha\hat{L}_y$, we have the following property:

- Let $|\chi(\alpha)\rangle_A$ and $|\phi(\alpha)\rangle_B$ be intelligent:

$$\begin{aligned}(\hat{L}_x - i\alpha\hat{L}_y)|\chi(\alpha)\rangle_A &= \lambda_A|\chi(\alpha)\rangle_A & (32)\\ (\hat{L}_x - i\alpha\hat{L}_y)|\phi(\alpha)\rangle_B &= \nu_B|\phi(\alpha)\rangle_B\,, & (33)\end{aligned}$$

- Then,

$$|\psi(\alpha)\rangle = |\chi\,(\alpha)\rangle_A \otimes |\phi\,(\alpha)\rangle_B \equiv |\chi\,(\alpha)\rangle_A\,|\phi\,(\alpha)\rangle_B \tag{34}$$

is intelligent.

Proof of compositeness

Define the total angular momentum projections as

$$\hat{L}_x = \hat{L}_{x,A} + \hat{L}_{x,B}\,, \qquad \hat{L}_y = \hat{L}_{y,A} + \hat{L}_{y,B} \tag{35}$$

where $\hat{L}_{x,A}$ acts only on $|\chi(\alpha)\rangle_A$ and not on $|\phi(\alpha)\rangle_B$, *etc.*

$$\begin{aligned}
(\hat{L}_x - i\alpha\hat{L}_y)|\psi(\alpha)\rangle \\
&= \left[(\hat{L}_{x,A} - i\alpha\hat{L}_{y,A})|\chi(\alpha)\rangle_A\right]|\phi(\alpha)\rangle_B \\
&\quad + |\chi(\alpha)\rangle_A\left[(\hat{L}_{x,B} - i\alpha\hat{L}_{y,B})|\phi(\alpha)\rangle_B\right] && (36) \\
&= (\lambda_A + \nu_B)|\chi(\alpha)\rangle_A|\phi(\alpha)\rangle_B. && (37)
\end{aligned}$$

In other words: the direct product of two intelligent states is also intelligent, provided that one thinks of the resulting state as a composite state constructed from two separate systems.

Intelligent states as coupled coherent states

- We can use the coupling property on coherent states (they are intelligent) to construct other intelligent (non-coherent) states.
- Example:

$$\left(\mathrm{e}^{-i\beta\hat{L}_{y,A}}|\ell_A,\ell_A\rangle\right)\left(\mathrm{e}^{-i\beta\hat{L}_{y,B}}|\ell_B,\ell_B\rangle\right) \tag{38}$$

is intelligent but...

- this is nothing new since

$$\begin{aligned}\left(\mathrm{e}^{-i\beta\hat{L}_{y,A}}|\ell_A,\ell_A\rangle\right)&\left(\mathrm{e}^{-i\beta\hat{L}_{y,B}}|\ell_B,\ell_B\rangle\right)\\ &=\left(\mathrm{e}^{-i\beta(\hat{L}_{y,A}+\hat{L}_{y,B})}\right)|\ell_A,\ell_A\rangle|\ell_B,\ell_B\rangle\,, &(39)\\ &=\mathrm{e}^{-i\beta\hat{L}_y}|\ell,\ell\rangle\,, &(40)\end{aligned}$$

with $\ell=\ell_A+\ell_B$.

Intelligent states as coupled coherent states

- Consider instead

$$|\psi_{\ell_A,\ell_B}(\beta)\rangle = \left(\mathrm{e}^{-i\beta\hat{L}_{y,A}}|\ell_A,\ell_A\rangle\right)\left(\mathrm{e}^{+i\beta\hat{L}_{y,B}}|\ell_B,\ell_B\rangle\right) . \quad (41)$$

- Since this is a product of intelligent states, it is also intelligent.
- Write

$$\begin{aligned} |\psi_{\ell_A,\ell_B}(\beta)\rangle &= \left[\sum_{m_A}|\ell_A,m_A\rangle\langle\ell_A,m_A|\mathrm{e}^{-i\beta\hat{L}_{y,A}}|\ell_A,\ell_A\rangle\right] \\ &\times \left[\sum_{m_B}|\ell_B,m_B\rangle\langle\ell_B,m_B|\mathrm{e}^{i\beta\hat{L}_{y,A}}|\ell_B,\ell_B\rangle\right] \end{aligned} \quad (42)$$

and couple the angular momentum states using

$$\mathbb{1} = \sum_m |\ell,m\rangle\langle\ell,m| \quad (43)$$

Intelligent states as coupled coherent states

- Using
$$\langle \ell, m|\mathrm{e}^{-i\beta\hat{L}_y}|\ell,\ell\rangle = d^{\ell}_{m,\ell}(\beta)\,, \tag{44}$$
this yields the unnormalized closed form expression:
$$|\psi^{\ell}_{\ell_A,\ell_B}(\beta)\rangle = \sum_m |\ell, m\rangle\, C^{\ell,m}_{\ell_A,m_A;\ell_B,m_B} \times d^{\ell_A}_{m_A,\ell_A}(\beta)\, d^{\ell_B}_{m_B,\ell_B}(-\beta) \tag{45}$$
where $C^{\ell,m}_{\ell_A,m_A;\ell_B,m_B}$ is an angular momentum coupling coefficient.
- The resulting states $|\psi^{\ell}_{\ell_A,\ell_B}(\beta)\rangle$ must be normalized "by hand",

Intelligent states as coupled coherent states

Observations:

- The eigenvalue problem

$$\left(\hat{L}_x - i\alpha\hat{L}_y\right)|\psi\rangle = \lambda|\psi\rangle \tag{46}$$

  has $2\ell + 1$ solution for angular momentum ℓ.
  - Choosing $(\ell_A, \ell_B) = (\ell, 0), (\ell - \frac{1}{2}, \frac{1}{2}), (\ell - 1, 1)$ *etc* yields the $2\ell + 1$ possible solutions.
  - One can show $\lambda = \sqrt{1 - |\alpha|^2}(\ell_A - \ell_B)$.
- Assuming $|\alpha| \leq 1$,
  - λ is real and $\lambda_{\ell_A,\ell_B} = (\ell_A - \ell_B)\sin\beta$,
  - $\langle\hat{L}_x\rangle = \frac{1}{2}(\ell_B - \ell_A)\sin\beta$
  - $\langle\hat{L}_y\rangle = 0$
- For $|\alpha| > 1$, one must project from

$$|\psi_{\ell_A,\ell_B}(\beta)\rangle = \left(\mathrm{e}^{-i\beta\hat{L}_{x,A}}|\ell_A,\ell_A\rangle\right)\left(\mathrm{e}^{i\beta\hat{L}_{x,B}}|\ell_B,\ell_B\rangle\right) \tag{47}$$

A possible implementation

- One can avoid the use of coupling technology by taking advantage of the bosonic nature of photons.
- It has been shown that one can write

$$
\begin{aligned}
&|\psi^{\ell}_{\ell_A,\ell_B}(\beta)\rangle \\
&\sim \left[a^\dagger_H \cos(\frac{\beta}{2}) + a^\dagger_V \sin(\frac{\beta}{2})\right]^{2\ell_A} \\
&\times \left[a^\dagger_H \cos(\frac{\beta}{2}) - a^\dagger_V \sin(\frac{\beta}{2})\right]^{2\ell_B} |0\rangle
\end{aligned} \tag{48}
$$

- Unfortunately, this is apparently very difficult to do in the lab.

A non-unitary transformation

Recall that intelligent states satisfy

$$\left(\hat{L}_x - i\alpha\hat{L}_y\right)|\psi^\ell\rangle = \lambda|\psi^\ell\rangle \tag{49}$$

- Consider

$$|\ell, m; \nu) \equiv \mathrm{e}^{i\pi\hat{L}_y/2}\mathrm{e}^{\nu\hat{L}_x}|\ell, m\rangle \tag{50}$$

- Note that, since $\hat{L}_x$ is hermitian, $\mathrm{e}^{\nu\hat{L}_x}$ is not unitary. Then

$$\begin{aligned}&\left(\hat{L}_x - i\alpha\hat{L}_y\right)\mathrm{e}^{i\pi\hat{L}_y/2}\mathrm{e}^{\nu\hat{L}_x}|\ell, m\rangle \\ &= \mathrm{e}^{i\pi\hat{L}_y/2}\mathrm{e}^{\nu\hat{L}_x} \\ &\quad\times\left[\mathrm{e}^{-\nu\hat{L}_x}\mathrm{e}^{-i\pi\hat{L}_y/2}\left(\hat{L}_x - i\alpha\hat{L}_y\right)\mathrm{e}^{i\pi\hat{L}_y/2}\mathrm{e}^{\nu\hat{L}_x}\right]|\ell, m\rangle\end{aligned} \tag{51}$$

- Using BCH:

$$\begin{aligned}&\left[\mathrm{e}^{-\nu\hat{L}_x}\mathrm{e}^{-i\pi\hat{L}_y/2}\left(\hat{L}_x - i\alpha\hat{L}_y\right)\mathrm{e}^{i\pi\hat{L}_y/2}\mathrm{e}^{\nu\hat{L}_x}\right] \\ &\qquad = (\cosh\nu + \alpha\sinh\nu)\,\hat{L}_z + i\,(\sinh\nu + \alpha\cosh\nu)\,\hat{L}_y\end{aligned}$$

A non-unitary transformation

- Choose ν so that

$$\sinh\nu + \alpha\cosh\nu = 0 \Rightarrow \tanh\nu = -\alpha\,. \tag{52}$$

- Then

$$\begin{aligned}\left(\hat{L}_x - i\alpha\hat{L}_y\right)&\left[\mathrm{e}^{i\pi\hat{L}_y/2}\mathrm{e}^{\nu\hat{L}_x}|\ell, m\rangle\right]\\ &= m\sqrt{1-\alpha^2}\,\left[\mathrm{e}^{i\pi\hat{L}_y/2}\mathrm{e}^{\nu\hat{L}_x}|\ell, m\rangle\right]\,.\end{aligned} \tag{53}$$

Compare with:

$$\left(\hat{L}_x - i\alpha\hat{L}_y\right)|\psi^{\ell}_{\ell_A,\ell_B}(\beta)\rangle = (\ell_B - \ell_A)\sin\beta\,|\psi^{\ell}_{\ell_A,\ell_B}(\beta)\rangle \tag{54}$$

- Recall $\cos\beta = -\alpha$ so $\sin\beta = \sqrt{1-\alpha^2}$.
- Thus $m = \ell_B - \ell_A$

Normalizing $|\ell, m; \nu)$

As the transformation is not unitary, the states $|\ell, m; \nu)$ are not normalized.

- Start with
$$\langle \ell, m| \mathrm{e}^{\nu \hat{L}_x} \mathrm{e}^{-i\pi \hat{L}_y/2} \mathrm{e}^{i\pi \hat{L}_y/2} \mathrm{e}^{\nu \hat{L}_x} |\ell, m\rangle = \langle \ell, m| \mathrm{e}^{2\nu \hat{L}_x} |\ell, m\rangle\,, \quad (55)$$
- Next, observe that
$$|\ell, m\rangle = \mathrm{e}^{-im\pi/2} \mathrm{e}^{i\pi \hat{L}_z/2} |\ell, m\rangle \quad (56)$$
so
$$\begin{aligned} \langle \ell, m| \mathrm{e}^{2\nu \hat{L}_x} |\ell, m\rangle &= \langle \ell, m| \mathrm{e}^{-i\pi \hat{L}_z/2} \mathrm{e}^{2\nu \hat{L}_x} \mathrm{e}^{i\pi \hat{L}_z/2} |\ell, m\rangle\,, && (57)\\ &= \langle \ell, m| \mathrm{e}^{2\nu \hat{L}_y} |\ell, m\rangle\,, && (58)\\ &= d^{\ell}_{m,m}(-2i\nu) && (59)\end{aligned}$$
where $d^{\ell}_{m,m}(-2i\nu)$ is the Wigner little-d functions with imaginary argument.
- Thus
$$|\ell, m; \nu\rangle = \frac{1}{\sqrt{|d^{\ell}_{m,m}(-2i\nu)|}} |\ell, m; \nu)\,. \quad (60)$$

Closed form expression

Putting it all together:

$$|\ell, m; \nu\rangle = \frac{1}{\sqrt{|d^{\ell}_{mm}(-2i\nu)|}} \mathrm{e}^{i\pi \hat{L}_y/2}\, \mathrm{e}^{\nu \hat{L}_x} |\ell, m\rangle\,, \tag{61}$$

which can be expanded as

$$\begin{aligned}
&|\ell, m; \nu\rangle \\
&\quad = \frac{1}{\sqrt{|d^{\ell}_{mm}(2i\nu)|}} \mathrm{e}^{i\pi \hat{L}_y/2}\, \mathrm{e}^{\nu \hat{L}_x} \mathrm{e}^{-i\pi \hat{L}_y/2} \left[\mathrm{e}^{i\pi \hat{L}_y/2} |\ell, m\rangle\right] \\
&\quad = \frac{1}{\sqrt{|d^{\ell}_{mm}(-2i\nu)|}}\, \mathrm{e}^{-\nu \hat{L}_z} \left[\sum_{m'} |\ell, m'\rangle d^{\ell}_{m'm}(-\pi/2)\right],
\end{aligned} \tag{62}$$

and finally gives the closed form expression

$$|\ell, m; \nu\rangle = \sum_{m'} \frac{d^{\ell}_{m'm}(-\pi/2)\, \mathrm{e}^{-\nu m'}}{\sqrt{|d^{\ell}_{mm}(-2i\nu)|}}\, |\ell, m'\rangle\,. \tag{63}$$

Summary

- Angular momentum coherent states are intelligent.
- Not all intelligent states are coherent.
- Two methods of construction:
  - Based on coupling and unitary transformations,
  - Based on non-unitary transformations but no coupling.

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